2008 Quarter-final Exam: Problem 4

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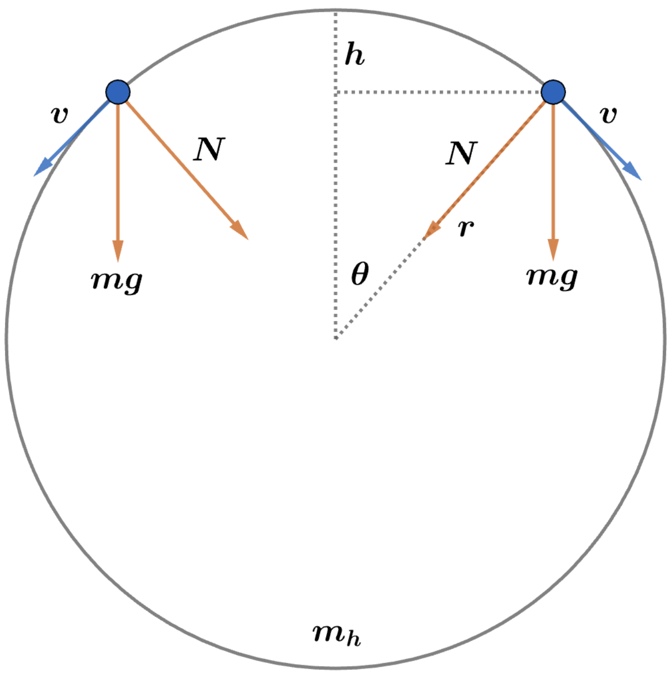


When the beads are at angle θ from the top, we can conserve energy to find their speed v (they have the same speed by symmetry),

$$mgh = mgr(1 - \cos\theta) = \frac{1}{2}mv^2$$

$$v^2 = 2gr(1 - \cos\theta)$$

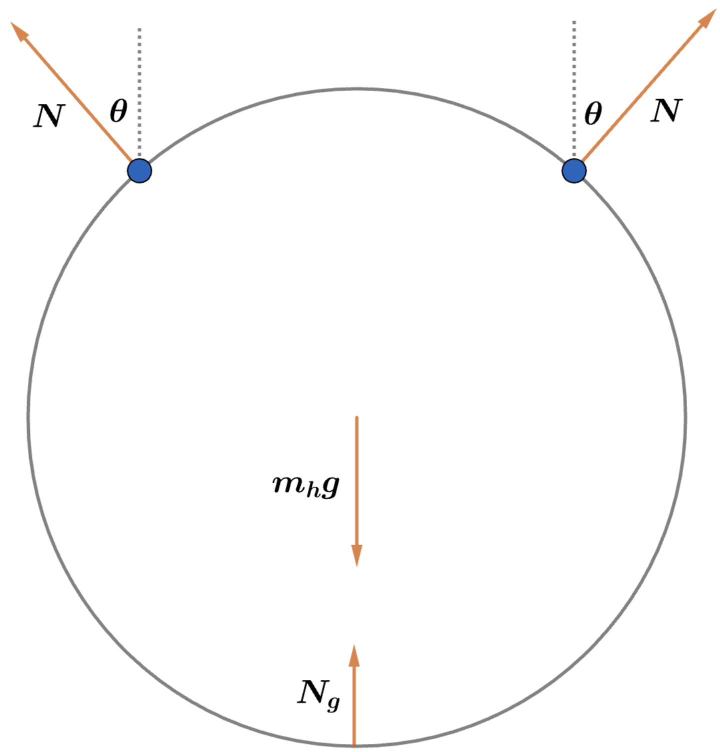
Applying Newton's 2nd law in the radial direction, we have

$$N + mg\cos\theta = \frac{mv^2}{r} = 2mg(1 - \cos\theta)$$

using our earlier expression for v. Solving for the normal force,

$$N = mg(2 - 3\cos\theta)$$

By Newton's 3rd law, the hoop feels the normal force directed in the opposite direction.



Balancing forces on the hoop,

$$N_g + 2N\cos\theta = m_h g$$

For the hoop to stay on the ground, we must have $N_g \geq 0$ so

$$m_h g - 2N\cos\theta \geq 0$$

$$2N\cos\theta \leq m_h g$$

Substituting our expression for N,

$$2mg(2 - 3\cos\theta)\cos\theta \leq m_h g$$

$$\frac{m}{m_h} \leq \frac{1}{2(2-3\cos\theta)\cos\theta}$$

To maximize the right-hand side, we can find the minimum of $(2 - 3\cos\theta)\cos\theta$ by differentiation:

$$\frac{d}{d\cos\theta}[(2-3\cos\theta)\cos\theta] = -3\cos\theta + (2 - 3\cos\theta) = 2 - 6\cos\theta = 0$$

Since $\cos\theta = 1/3$, we find $(2 - 3\cos\theta)\cos\theta = 1/3$ so

$$\boxed{\frac{m}{m_h} \leq \frac{1}{2(1/3)} = \frac{3}{2}}$$