2023 F=ma Exam: Problem 21

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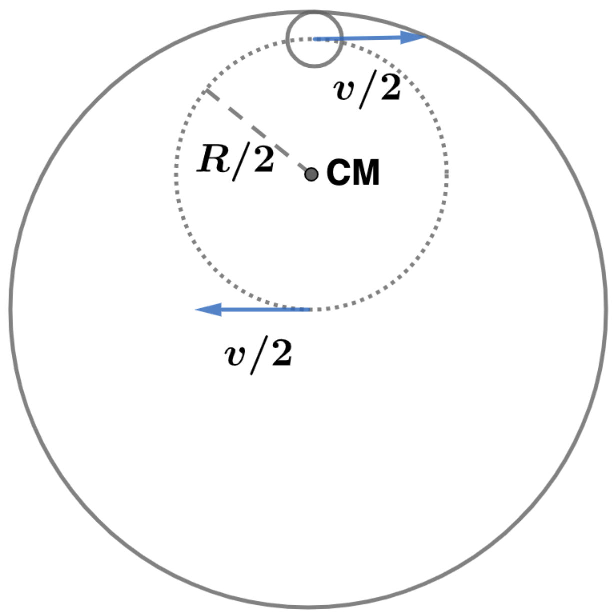
There is no friction and there are no external forces or torques on the system so energy, linear momentum, and angular momentum are all conserved. We first go to the CM frame of the system where the mass and ring are initially moving with speed $v/2$ in opposite directions.



By conservation of linear momentum, we have

$$m\vec{v}_{\text{ring}} + m\vec{v}_{\text{mass}} = \vec{p}_i = 0$$
$$\vec{v}_{\text{ring}} = -\vec{v}_{\text{mass}}$$

By conservation of energy, we have

$$\frac{1}{2}mv_{\text{ring}}^2 + \frac{1}{2}mv_{\text{mass}}^2 = E_i = \frac{1}{2}m\left(\frac{v}{2}\right)^2 + \frac{1}{2}m\left(\frac{v}{2}\right)^2$$

so $|\vec{v}_{\text{mass}}| = |\vec{v}_{\text{ring}}| = v/2$. Thus, the mass and ring always move in opposite directions with speed $v/2$.

Both objects have the same mass so they are at the same distance r on opposite sides of the CM. Using conservation of angular momentum around the CM, we have

$$m\left(\frac{v}{2}\right)r\sin\theta + m\left(\frac{v}{2}\right)r\sin\theta = L_i = m\left(\frac{v}{2}\right)\left(\frac{R}{2}\right) + m\left(\frac{v}{2}\right)\left(\frac{R}{2}\right)$$

$$r\sin\theta = \frac{R}{2}$$

so $r \geq R/2$. Since the mass is inside the ring, this constrains the mass to be at most a distance $R/2$ from their CM so we also have $r \leq R/2$. Hence, $r = R/2$ for all time and we conclude the two objects are in a circular orbit around their CM.

The time it takes for one orbit is then

$$T = \frac{2\pi(R/2)}{(v/2)} = \frac{2\pi R}{v}$$

so the answer is $\boxed{\text{C}}$.