2022B F=ma Exam: Problem 25

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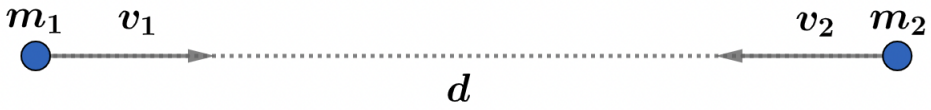

For two masses m_1, m_2 falling towards each other under gravity, starting at separation distance r, we have

$$E = -\frac{Gm_1m_2}{r} = -\frac{Gm_1m_2}{d} + \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$$

by conservation of energy. There are no external forces so linear momentum is also conserved:

$$m_1v_1 = m_2v_2$$

Combining both equations,

$$Gm_1m_2\left(\frac{1}{d} - \frac{1}{r}\right) = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2\left(\frac{m_1v_1}{m_2}\right)^2 = \frac{1}{2}\left(\frac{m_1m_2}{m_2} + \frac{m_1^2}{m_2}\right)v_1^2$$

$$Gm_2\left(\frac{1}{d} - \frac{1}{r}\right) = \frac{(m_1+m_2)}{2m_2}v_1^2$$

Thus,

$$v_1 = m_2\sqrt{\frac{2G}{m_1+m_2}\left(\frac{1}{d} - \frac{1}{r}\right)}$$

$$v_2 = m_1\sqrt{\frac{2G}{m_1+m_2}\left(\frac{1}{d} - \frac{1}{r}\right)}$$

Their relative speed is

$$v_r = v_1 + v_2 = \sqrt{2G(m_1+m_2)\left(\frac{1}{d} - \frac{1}{r}\right)} \propto \sqrt{m_1+m_2}$$

Since $T \sim v_r^{-1}$,

$$T \propto (m_1+m_2)^{-1/2}$$

$$\frac{T'}{T} = \sqrt{\frac{m+2m}{2m+2m}} = \sqrt{\frac{3}{4}}$$

so the answer is $\boxed{\text{D}}$.