A Modern Approach to Quantum Mechanics by Townsend - Solutions

Solutions by: GT SPS

Contributors: Charles Cardot, Cody McCord, Robin Glefke, Kiernan Reising, Kevin Jin, Alexander George-Kennedy, William Benston

Last Updated: January 3, 2021

Contents

10 Bound States of Central Potentials

10.1

We will only consider the radial portion of the wavefunction. This follows from chapter 9, where we saw that we can normalize the radial and angular portions of the wavefunction separately (see equation 9.138). Because of this, we need to include the factor of r^2, which arises from the Jacobian in spherical coordinates.

$$\begin{aligned}
\langle\psi|\,\hat{p}_r\,|\psi\rangle &= \frac{\hbar}{i}\int_0^\infty r^2 R^*(r)\left(\frac{d}{dr}+\frac{1}{r}\right)R(r)dr \\
&= \frac{\hbar}{i}\int_0^\infty r u^*(r)\left(\frac{d}{dr}+\frac{1}{r}\right)\frac{u(r)}{r}dr \\
&= \frac{\hbar}{i}\int_0^\infty u^*(r)u'(r)dr \\
&= \frac{\hbar}{i}\left[\cancelto{0}{u^*(\infty)u(\infty)} - u^*(0)u(0) - \int_0^\infty u^{*\prime}(r)u(r)dr\right] \\
&= i\hbar\left(|u(0)|^2 + \int_0^\infty u^{*\prime}(r)u(r)dr\right)
\end{aligned}$$

From here it's also easy to see that

$$\langle\psi|\,\hat{p}_r\,|\psi\rangle^* = -i\hbar\left(|u(0)|^2 + \int_0^\infty u'(r)u^*(r)dr\right)$$

Setting these two equations equal to one another and rearranging, we get

$$2|u(0)|^2 = -\int_0^\infty u^{*\prime}(r)u(r)dr - \int_0^\infty u'(r)u^*(r)dr$$

Once again, we can use integration by parts on one of these expressions. If we choose the second, then we get

$$\int_0^\infty u^*(r)u'(r)dr = -|u(0)|^2 - \int_0^\infty u^{*\prime}(r)u(r)dr$$

If we make the substitution above and rearrange the expression, we easily see that $|u(0)|^2 = 0$. This is only true when $u^*(0) = u(0) = 0$.

10.2

The Hamiltonian (page 348, equation 10.14) and energy eigenvalues (page 351, equation 10.34) for this system are

$$\hat{H} = \frac{\hat{p}^2}{2\mu} - \frac{e^2}{|\hat{r}|} \quad \text{and} \quad E_n = -\frac{\mu c^2\alpha^2}{2n^2}, \quad n = 1, 2, 3, ...$$

It's probably better to calculate $|\psi(t)\rangle$ first and then $\langle E\rangle$.

$$|\psi(t)\rangle = \frac{4}{5}e^{-iE_1t/\hbar}|1,0,0\rangle + \frac{3i}{5}e^{-iE_2t/\hbar}|2,1,1\rangle$$

$$\begin{aligned}\langle E\rangle &= \left|\frac{4}{5}e^{-iE_1t/\hbar}\right| E_1 + \left|\frac{3i}{5}e^{-iE_2t/\hbar}\right| E_2 \\ &= \frac{16E_1+9E_2}{25}\end{aligned}$$

Recall that $\hat{L}^2|n,l,m\rangle = l(l+1)\hbar^2|n,l,m\rangle$ and $\hat{L}_z|n,l,m\rangle = m\hbar|n,l,m\rangle$.

$$\hat{L}^2|\psi(t)\rangle = \frac{6i\hbar^2}{5}e^{-iE_2t/\hbar}|2,1,1\rangle \quad \text{and} \quad \hat{L}_z|\psi(t)\rangle = \frac{3i\hbar}{5}e^{-iE_2t/\hbar}|2,1,1\rangle$$

The expectation values of $\hat{L}^2$ and $\hat{L}_z$ are the inner products of $\langle\psi(t)|$ with $\hat{L}^2|\psi(t)\rangle$ and $\hat{L}_z|\psi(t)\rangle$.

$$\langle\psi(t)|\hat{L}^2|\psi(t)\rangle = \frac{18}{25}\hbar^2 \quad \text{and} \quad \langle\psi(t)|\hat{L}_z|\psi(t)\rangle = \frac{9}{25}\hbar$$

We also notice that none of these expectation values vary with time. This is expected, as all of these observables are eigenstates of the Hamiltonian. Observables which are not eigenstates of the Hamiltonian (such as $\hat{L}_x$ and $\hat{L}_y$) *will* have expectation values which vary with time.

10.3

a)

$$\begin{aligned}|\Psi(t)\rangle &= e^{-\frac{it}{\hbar}\left(\frac{\hat{p}^2}{2\mu}-\frac{e^2}{|\hat{r}|}+\omega_0\hat{L}_z\right)}|\Psi(0)\rangle \\ &= \frac{e^{\frac{-itE_1}{\hbar}}}{2}|1,0,0\rangle + \frac{e^{\frac{-it(E_2+\omega_0\hbar)}{\hbar}}}{\sqrt{2}}|2,1,1\rangle + \frac{e^{\frac{-itE_2}{\hbar}}}{2}|2,1,0\rangle \\ \left\langle\Psi(t)|\hat{H}|\Psi(t)\middle|\Psi(t)|\hat{H}|\Psi(t)\right\rangle &= \left|\frac{e^{\frac{-itE_1}{\hbar}}}{2}\right|^2 E_1 + \left|\frac{e^{\frac{-it(E_2+\omega_0\hbar)}{\hbar}}}{\sqrt{2}}\right|^2(E_2+\omega_0\hbar) + \left|\frac{e^{\frac{-itE_2}{\hbar}}}{2}\right|^2 E_2 \\ E &= \frac{E_1+2\omega_0\hbar+3E_2}{4}\end{aligned}$$

b)

$$\begin{aligned}\hat{L}_z &= \left\langle\Psi(t)|\hat{L}_z|\Psi(t)\middle|\Psi(t)|\hat{L}_z|\Psi(t)\right\rangle = \frac{\hbar}{2} \\ \hat{L}_x &= \left\langle\Psi(t)|\frac{\hat{L}_++\hat{L}_-}{2}|\Psi(t)\middle|\Psi(t)|\frac{\hat{L}_++\hat{L}_-}{2}|\Psi(t)\right\rangle \\ &= \frac{\sqrt{2}\hbar}{2}\left(\frac{1}{2\sqrt{2}}e^{it\omega_0}+\frac{1}{2\sqrt{2}}e^{-it\omega_0}\right) = \frac{\hbar}{2}\cos\omega_0 t\end{aligned}$$

10.4

The classically forbidden region, is when at some radius the potential energy exceeds the total energy, so first, we must find the radius at which the potential energy is equal to the ground state energy and more favorably in terms of the Bohr radius (a_0). **Note e in this first part is the charge of the electron.**

$$a_0 = \frac{\hbar}{\mu c \alpha}$$

$$\alpha = \frac{e^2}{4\pi\epsilon_0 c}$$

Subbing in α :

$$a_0 = \frac{\hbar}{\mu c \alpha} = \frac{4\pi\epsilon_0\hbar^2}{\mu e^2}$$

Solving for the radius :

$$\begin{aligned}
&E = V(r) = -\frac{\mu c^2}{2}\left(\frac{e^2}{4\pi\hbar\epsilon_0 c}\right)^2 = -\frac{e^2}{4\pi\epsilon_0 r} \\
&r\frac{\mu c^2}{2}\left(\frac{e^2}{4\pi\hbar\epsilon_0 c}\right)^2 = \frac{e^2}{4\pi\epsilon_0} \\
&r\frac{\mu c^2}{2} = \left(\frac{4\pi\hbar\epsilon_0 c}{e^2}\right)^2 \frac{e^2}{4\pi\epsilon_0} \\
&r\frac{\mu c^2}{2} = \frac{4\pi\hbar^2\epsilon_0 c^2}{e^2} \\
&r = 2\left(\frac{4\pi\epsilon_0\hbar^2}{\mu e^2}\right) = 2a_0
\end{aligned}$$

Now that we have the radius, we can find the probability by integrating from $2a_0$ to ∞ by using the equation below.

$$\int_{2a_0}^{\infty} r^2 R^*_{E,l} R_{E,l} dr$$

We already know the ground state wave equation from the book; we will plug that in for $R_{E,l}$. **Note e here is the exponential not the charge of the electron.**

$$\int_{2a_0}^{\infty} r^2 R^*_{E,l} R_{E,l} dr = \frac{4}{a_0^3}\int_{2a_0}^{\infty} e^{(\frac{-2r}{a_0})} r^2 dr$$

Integrating by parts gives us the result below.

$$\frac{4}{a_0^3}\left[-\frac{-a_0}{2}e^{(\frac{-2r}{a_0})}r^2 - \frac{a_0^2}{2}e^{(\frac{-2r}{a_0})}r - \frac{a_0^3}{4}e^{(\frac{-2r}{a_0})}\right]_{2a_0}^{\infty}$$
$$\frac{4}{a_0^3}\left[2a_0^3e^{-4} + a_0^3e^{-4} + \frac{a_0}{4}e^{-4}\right]$$

The probability is below.

$$13e^{-4}$$

10.5

$M_N = m_{proton} + m_{\text{neutron}} = 3.34 * 10^{-27}\ kg$, $m_e = 9.109 * 10^{-31}\ kg$, $m_{positron} = 9.109 * 10^{-31}\ kg$, $m_{muon} = 1.883 * 10^{-28}\ kg$, $c = 3 * 10^8\ m/s$, $\alpha = 1/137$, $\hbar = 6.5821 * 10^{-16}\ eV \cdot s$, $Z = 1$, note: $e^2 = \frac{e^2}{4\pi\epsilon_0}$

a)

$$E_n = -\frac{\mu Z^2 e^4}{2\hbar^2 n^2}$$
$$E_1 = -\frac{c^2 Z^2 \alpha^2}{2} \cdot \mu = -2.397 * 10^{12}\left(\frac{M_N m_e}{M_N + m_e}\right) = -2.18 * 10^{-18}\ J = -13.625\ eV$$
$$a_0 = \frac{\hbar}{\mu c \alpha} = 0.517\ \text{Angstrom}$$

b)

$$E_1 = -\frac{c^2 Z^2 \alpha^2}{2} \cdot \mu = -2.397 * 10^{12}\left(\frac{m_{positron} m_e}{m_{positron} + m_e}\right) = -1.092 * 10^{-18}\ J = -6.819\ eV$$
$$a_0 = \frac{\hbar}{\mu c \alpha} = 1.057\ \text{Angstrom}$$

c)

$$E_1 = -\frac{c^2 Z^2 \alpha^2}{2} \cdot \mu = -2.397 * 10^{12}\left(\frac{m_{proton} m_{muon}}{m_{proton} + m_{muon}}\right) = -4.056 * 10^{-16}\ J = -2531.96\ eV$$
$$a_0 = \frac{\hbar}{\mu c \alpha} = 0.0028\ \text{Angstrom}$$

d) Swap out Coulomb force $\frac{Ze^2}{r^2}$ for gravitational force $\frac{Gm_1m_2}{r^2}$, where $m_1 = m_2 = m_{neutron}$.

$$E_n = -\frac{\mu Z^2 e^4}{2\hbar^2 n^2} = \frac{\mu G^2 m_{neutron}^4}{2\hbar n^2}, \quad \mu = \frac{1}{2}m_{neutron}$$
$$E_1 = 1.37 * 10^{-121}\ J = 8.559 * 10^{-103}\ eV$$
$$a_0 = \frac{\hbar}{\mu c\alpha} = \frac{\hbar^2}{\mu e^2} = \frac{\hbar^2}{\mu G m_{neutron}^2} = 7.148 * 10^{22}\ m$$

10.6

10.7

10.8

We will start by solving the Schrodinger equation for the allowed energies $E < -V_0$.

$$\frac{-\hbar^2}{2\mu}\frac{d^2u}{dr^2} - V_0 u = Eu \qquad r < a$$
$$\frac{-\hbar^2}{2\mu}\frac{d^2u}{dr^2} = Eu \qquad r < a$$

This can now be rearranged and expressed in the form below.

$$\frac{d^2u}{dr^2} = \frac{-2\mu}{\hbar^2}(V_0 + E)u = -k^2 u \qquad r < a$$
$$\frac{d^2u}{dr^2} = \frac{-2\mu E}{\hbar^2}u = b^2 u \qquad r > a$$

We have defined k and b below.

$$k = \sqrt{\frac{2\mu}{\hbar^2}(V_0 + E)}$$
$$b = \sqrt{-\frac{2\mu E}{\hbar^2}}$$

This tells us that for the interval of $r < a$ our general solution will take the form of a superposition of sines and cosines, and for the interval $r > a$ our solution will take the form of the form of a superposition of exponential functions.

$$\begin{aligned} u &= Asin(kr) + Bcos(kr) & r < a \\ u &= Ce^{-br} + De^{br} & r > a \end{aligned}$$

The boundary conditions and the normalization condition give us the solutions below.

$$\begin{aligned} u &= Asin(kr) & r < a \\ u &= Ce^{-br} & r > a \end{aligned}$$

Now that we have the solutions to the Schrodinger equation, Lets consider the case $E < -V_0$.

10.9

The Hamiltonian for this system is given by

$$\hat{H} = \frac{\hat{p}}{2m} + V(x)$$

where the potential is given in the problem statement. This means we have two regions to consider so we must have two forms of the general wavefunction.

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - V_0\psi = E\psi \quad \text{when} \quad & -a > x > a \\ -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi \quad \text{when} \quad & -a < x < a \end{aligned}$$

To make the equations a little simpler, we can rearrange terms and combine them into constants.

$$\begin{aligned} \frac{d^2\psi}{dx^2} &= -k_0^2\psi, \quad \text{where} \quad k_0 = \sqrt{\frac{2m}{\hbar^2}(E + V_0)} \\ \frac{d^2\psi}{dx^2} &= q^2\psi, \quad \text{where} \quad q = \sqrt{\frac{-2mE}{\hbar^2}} \end{aligned}$$

The general form for the wave equation is an oscillating wave inside of the well and exponentially decaying outside.

$$\psi = A\sin(k_0 x) + B\cos(k_0 x) \quad \text{when} \quad -a < x < a$$
$$\psi = Ce^{qx} \quad \text{when} \quad x < -a$$
$$\psi = De^{-qx} \quad \text{when} \quad x > a$$

At the boundaries the solutions *and* their derivatives must have the same values. At x = a we have

$$De^{-qa} = A\sin(k_0 a) + B\cos(k_0 a$$
$$\frac{-qD}{2}e^{-qa} = \frac{Ak_0}{2}\cos(k_0 a) - \frac{Bk_0}{2}\sin(k_0 a)$$

At x = -a we have

$$Ce^{-qa} = -A\sin(k_0 a) + B\cos(k_0 a$$
$$\frac{-qC}{2}e^{-qa} = -\frac{Ak_0}{2}\cos(k_0 a) - \frac{Bk_0}{2}\sin(k_0 a)$$

When the oscillating term is symmetric, A = 0 and D = C.

$$B\cos(k_0 a) = Ce^{-qa} \quad \text{and} \quad Bk_0\sin k_0 a = qCe^{-qa} \quad \overset{divide}{\rightarrow} \quad k_0\tan(k_0 a) = q$$

When the oscillating term is anti-symmetric, B = 0 and D = -C.

$$A\sin(k_0 a) = -Ce^{-qa} \quad \text{and} \quad Ak_0\cos k_0 a = qCe^{-qa} \quad \overset{divide}{\rightarrow} \quad k_0\cot(k_0 a) = -q$$

If we make the quick substitutions of $\xi = k_0 a$ and $\eta = qa$, then we have the two simple equations

$$\xi\tan(\xi) = \eta \quad \text{and} \quad -\xi\cot(\xi) = \eta$$

From these constraints we see that ξ must be periodic in the range 0 to $\pi/2$. It's also interesting to note that the sum of the squares of ξ and η must be a constant *that is independent of E*. It's easy to see that this extra constraint is simply a circle centered at the origin.

$$0 < \xi < \frac{\pi}{2}$$
$$\xi^2 + \eta^2 = r_0^2, \quad \text{where} \quad r_0 = \sqrt{\frac{mV_0L^2}{2\hbar^2}}$$

Bound state solutions exist where the tangent and/or cotangent functions intersect with the circle created from ξ and η. We can graphically resent this and show that, for any potential V_0, the circle will *always* intersect at least one function (figure 1). We see that the lowest energy bound state will be a symmetric wavefunction.

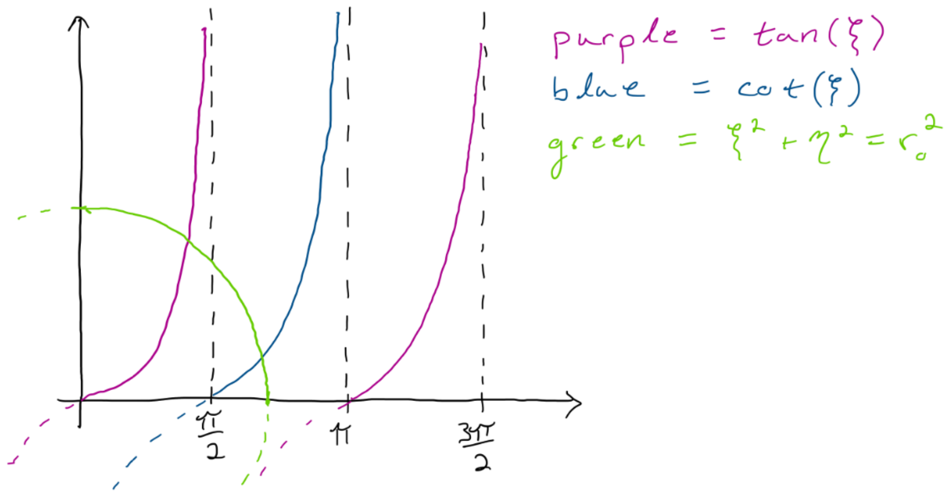

Figure 1: Purple curves represent symmetric wavefunctions and blue curves represent antisymmetric wavefunctions. The radius of the green circle is proportional to the size of the potential.

10.10

Inside the well:

$$-\frac{\hbar^2}{2\mu}\left(\frac{\partial}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\psi(x,y,z)=E\psi(x,y,z). \tag{1}$$

We assume the wavefunction is separable

$$\psi(x,y,z)=X(x)Y(y)Z(z).$$

$$X_{xx}+k_x^2X(x)=0 \qquad k_x^2=\frac{2\mu E_x}{\hbar^2} \tag{2}$$

$$Y_{yy}+k_y^2Y(y)=0 \qquad k_y^2=\frac{2\mu E_y}{\hbar^2} \tag{3}$$

$$Z_{zz}+k_z^2Z(z)=0 \qquad k_z^2=\frac{2\mu E_z}{\hbar^2} \tag{4}$$

General solutions:

$$\begin{aligned} X(x)&=\alpha_x\sin k_xx+\beta_x\cos k_xx\\ Y(Y)&=\alpha_y\sin k_yy+\beta_y\cos k_yy\\ Z(z)&=\alpha_z\sin k_zz+\beta_z\cos k_zz \end{aligned}$$

Boundary conditions:

$$\psi(0,y,z)=\psi(a,y,z)=0 \tag{5}$$
$$\psi(x,0,z)=\psi(x,a,z)=0 \tag{6}$$
$$\psi(x,y,0)=\psi(x,y,a)=0. \tag{7}$$

$$X(x)=\sqrt{\frac{2}{a}}\sin k_x x \tag{8}$$
$$Y(y)=\sqrt{\frac{2}{a}}\sin k_y y \tag{9}$$
$$Z(z)=\sqrt{\frac{2}{a}}\sin k_z z, \tag{10}$$

where

$$k_x=\frac{n_x\pi}{a} \qquad n_x=1,2,\ldots \tag{11}$$
$$k_y=\frac{n_y\pi}{a} \qquad n_y=1,2,\ldots \tag{12}$$
$$k_z=\frac{n_z\pi}{a} \qquad n_z=1,2,\ldots. \tag{13}$$

Thus

$$\psi(x,y,z)=\sqrt{\frac{8}{a^3}}\sin\frac{n_x\pi}{a}x\sin\frac{n_y\pi}{a}y\sin\frac{n_z\pi}{a}z$$
$$E_{n_x,n_y,n_z}=E_x+E_y+E_z=\frac{\hbar^2\pi^2}{2\mu a^2}(n_x^2+n_y^2+n_z^2).$$

Volume of a sphere: $\frac{4}{3}\pi a^3$ vs Volume of a cube: a^3

$$\hat{H}=\hat{H}_x+\hat{H}_y+\hat{H}_z=\frac{\hat{p}_x^2}{2\mu}+V(x)+\frac{\hat{p}_y^2}{2\mu}+V(y)+\frac{\hat{p}_z^2}{2\mu}+V(z)$$
$$E=\frac{\hbar^2\pi^2}{2\mu a^2}(n_x^2+n_y^2+n_z^2);\ n_x,n_y,n_z=1,2,3,\ ...$$
$$E_0=\frac{3\hbar^2\pi^2}{2\mu a^2}\text{ cube}$$
$$E_0=\frac{\hbar^2\pi^2}{2\mu a^2}\text{ sphere}$$

10.11

10.12

a)

$$u = \rho^{l+1} e^{\frac{-\rho^2}{2}} f(\rho) \text{ where } f(\rho) = \sum_{k=0}^{\infty} c_k \rho^k$$

$$\frac{du}{d\rho} = (l+1)\rho^l (e^{\frac{-\rho^2}{2}} f(\rho)) + \rho^{l+1} \left(-\rho e^{\frac{-\rho^2}{2}} f(\rho) + e^{\frac{-\rho^2}{2}} \frac{df}{d\rho} \right)$$

$$\frac{d^2u}{d\rho^2} = \rho^{l+1} e^{\frac{-\rho^2}{2}} \frac{d^2 f}{d\rho^2} + \left(2(l+1)\rho^l e^{\frac{-\rho^2}{2}} - 2\rho^{l+2} e^{\frac{-\rho^2}{2}} \right) \frac{df}{d\rho}$$

$$+ e^{\frac{-\rho^2}{2}} \left(l(l+1)\rho^{l-1} - 2(l+1)\rho^{l+1} - \rho^{l+1} + \rho^2 \rho^{l+1} \right) f(\rho)$$

$$\frac{d^2u}{d\rho^2} - \frac{l(l+1)}{\rho^2} u - \rho^2 u = -\lambda u \tag{10.91}$$

Plug above expressions for u, $\frac{du}{d\rho}$, and $\frac{d^2u}{d\rho^2}$ into the expression (10.91)

$$\rho^{l+1} e^{\frac{-\rho^2}{2}} \frac{d^2 f}{d\rho^2} + \left(2(l+1)\rho^l e^{\frac{-\rho^2}{2}} - 2\rho^{l+2} e^{\frac{-\rho^2}{2}} \right) \frac{df}{d\rho}$$

$$+ e^{\frac{-\rho^2}{2}} \left(l(l+1)\rho^{l-1} - 2(l+1)\rho^{l+1} - \rho^{l+1} + \rho^2 \rho^{l+1} \right) f(\rho)$$

$$- \frac{l(l+1)}{\rho^2} \left((l+1)\rho^l (e^{\frac{-\rho^2}{2}} f(\rho)) + \rho^{l+1} \left(-\rho e^{\frac{-\rho^2}{2}} f(\rho) + e^{\frac{-\rho^2}{2}} \frac{df}{d\rho} \right) \right)$$

$$- \rho^2 \rho^{l+1} e^{\frac{-\rho^2}{2}} f(\rho) = -\lambda \rho^{l+1} e^{\frac{-\rho^2}{2}} f(\rho)$$

Cancel all $e^{\frac{-\rho^2}{2}}$, simplify equation, move λ coefficient to the other side and set equal to 0.

$$\rho \frac{d^2 f}{d\rho^2} + 2(l+1-\rho^2)\frac{df}{d\rho} + (-\rho - 2\rho(l+1) + \lambda\rho) f(\rho) = 0$$

b) Apply the Frobenius method, https://www.youtube.com/watch?v=SS6bniyB7rw

$$f(\rho) = \sum_{k=0}^{\infty} c_k \rho^k, \quad \frac{df}{d\rho} = \sum_{k=1}^{\infty} k c_k \rho^{k-1}, \quad \frac{d^2 f}{d\rho^2} = \sum_{k=2}^{\infty} (k-1)k c_k \rho^{k-2}$$

Substitute in expressions for f, $\frac{df}{d\rho}$, and $\frac{d^2 f}{d\rho^2}$, and solve for coefficients that will force the equation on the left hand side (of the final expression given in part a) of this problem) to be zero. This results in a two-term recursion relationship:

$$c_{k+2} = \frac{2k + 1 + 2(l+1) - \lambda}{(k+2)(k+1) + 2(l+1)(k+2)} c_k$$

Solve for the condition in which the two-term recursion relationship is zero.

$$\lambda = 2k + 2l + 3 = \frac{2E}{\hbar\omega} \quad \text{From equation 10.90 in the textbook}$$
$$E = (k + l + \frac{3}{2})\hbar\omega = (2n_r + l + \frac{3}{2})\hbar\omega$$

Because only even k's are left after the satisfying the coefficient requirements given by the Frobenius method, k = 0, 2, 4, ... can be rewritten as $2n_r$ where $n_r = 0, 1, 2, ...$

10.13

10.14

10.15

10.16

10.17

a) **Show that the energy eigenvalues are given by $E_n = (n+1)\hbar\omega$, where the integer $n = n_1 + n_2$, with $n_1,\ n_2 = 0, 1, 2, \ldots$**

This Hamiltonian is the sum of the x and y components, whose eigenenergies are

$$E_{n_i} = \left(n_i + \frac{1}{2}\right)\hbar\omega \qquad \text{for } i = x, y, \ldots \qquad \text{and } n = 0, 1, 2, \ldots$$

Therefore, when we sum them together we get

$$E_n = \left(n_x + \frac{1}{2}\right)\hbar\omega + \left(n_y + \frac{1}{2}\right)\hbar\omega = (n_x + n_y + 1)\,\hbar\omega$$

Which shows $E_n = (n+1)\hbar\omega$, where the integer $n = n_1 + n_2$, with $n_1,\ n_2 = 0, 1, 2, \ldots$

b) **Express the operator $\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$ in terms of the lowering operators**

$$\hat{a}_1 = \sqrt{\frac{\mu\omega}{2\hbar}}\left(\hat{x} + \frac{i}{\mu\omega}\hat{p}_x\right) \qquad \textit{and} \qquad \hat{a}_2 = \sqrt{\frac{\mu\omega}{2\hbar}}\left(\hat{y} + \frac{i}{\mu\omega}\hat{p}_y\right)$$

and the corresponding raising operators $\hat{a}_1^\dagger$ and $\hat{a}_2^\dagger$. Give a symmetry argument showing $[\hat{H}, \hat{L}_z] = 0$. Evaluate this commutator directly and confirm that it indeed vanishes.

From eq 7.11 and 7.12 in Townsend, we know we can rewrite the raising and lowering operators as

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger) \qquad \hat{p}_x = -i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a} - \hat{a}^\dagger)$$

and $\hat{y}$ and $\hat{p}_y$ are similar. Therefore, substituting them into the $\hat{L}_z$ commutation relation

$$\begin{aligned}\hat{L}_z &= \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_1 + \hat{a}_1^\dagger)(-i)\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}_2 - \hat{a}_2^\dagger)\\ &\quad -\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_2 + \hat{a}_2^\dagger)(-i)\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}_1 - \hat{a}_1^\dagger)\\ &= -i\frac{\hbar}{2}(\hat{a}_1 + \hat{a}_1^\dagger)(\hat{a}_2 - \hat{a}_2^\dagger) + i\frac{\hbar}{2}(\hat{a}_2 + \hat{a}_2^\dagger)(\hat{a}_1 - \hat{a}_1^\dagger)\\ &= -i\frac{\hbar}{2}(\hat{a}_1\hat{a}_2 - \hat{a}_1\hat{a}_2^\dagger + \hat{a}_1^\dagger\hat{a}_2 - \hat{a}_1^\dagger\hat{a}_2^\dagger) + i\frac{\hbar}{2}(\hat{a}_2\hat{a}_1 - \hat{a}_2\hat{a}_1^\dagger + \hat{a}_2^\dagger\hat{a}_1 - \hat{a}_2^\dagger\hat{a}_1^\dagger)\end{aligned}$$

Note that $\hat{a}_2$ and $\hat{a}_1$ both commute, as do their respective raising operators. Therefore

$$\hat{L}_z = i\frac{\hbar}{2}(\hat{a}_1\hat{a}_2^\dagger - \hat{a}_1^\dagger\hat{a}_2 - \hat{a}_2\hat{a}_1^\dagger + \hat{a}_2^\dagger\hat{a}_1) = i\hbar(\hat{a}_1\hat{a}_2^\dagger - \hat{a}_1^\dagger\hat{a}_2)$$

If we convert the Hamiltonian into cylindrical coordinates, it is clear that in the x-y plane (or θ-plane), the Hamiltonian is constant following a rotation about the z-axis because there is no θ dependence. Because of this z-rotation symmetry, the Hamiltonian must commute with $\hat{L}_z$. We can also explicitly show this by evaluating the commutator with the original representation of $\hat{L}_z$

$$\begin{aligned}[\hat{H}, \hat{L}_z] &= [\frac{\hat{p}_x^2}{2\mu} + \frac{1}{2}\mu\omega^2\hat{x}^2 + \frac{\hat{p}_y^2}{2\mu} + \frac{1}{2}\mu\omega^2\hat{y}^2, \hat{x}\hat{p}_y - \hat{y}\hat{p}_x]\\ &= [\frac{\hat{p}_x^2}{2\mu}, \hat{x}\hat{p}_y] + [\frac{1}{2}\mu\omega^2\hat{x}^2, \hat{x}\hat{p}_y] + [\frac{\hat{p}_y^2}{2\mu}, \hat{x}\hat{p}_y] + [\frac{1}{2}\mu\omega^2\hat{y}^2, \hat{x}\hat{p}_y] - [\frac{\hat{p}_x^2}{2\mu}, \hat{y}\hat{p}_x]\\ &\quad -[\frac{1}{2}\mu\omega^2\hat{x}^2, \hat{y}\hat{p}_x] - [\frac{\hat{p}_y^2}{2\mu}, \hat{y}\hat{p}_x] - [\frac{1}{2}\mu\omega^2\hat{y}^2, \hat{y}\hat{p}_x] = 0\end{aligned}$$

Recall that $\hat{p}_x$ and $\hat{p}_y$ are of the same form/magnitude because this is a homogeneous oscillator. Therefore corresponding $\hat{y}\hat{p}_x$ and $\hat{x}\hat{p}_y$ terms in the expanded commutator will cancel, producing a zero result and proving commutation.

c) **Determine the correct linear combination of the energy eigenstates with energy $E_1 = 2\hbar\omega$ that are eigenstates of $\hat{L}_z$ by diagonalizing the matrix representation of $\hat{L}_z$ restricted to this subspace of states.**

$$E_1 = 2\hbar\omega,\ (n_x = 1\ \ n_y = 0),\ (n_x = 0\ \ n_y = 1)$$

Given the restrictions on the possible values of n_x and n_y, the raising and lowering operators must take the form of

$$\hat{a}^\dagger = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \hat{a} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

meaning that $\hat{L}_z = i\hbar(\hat{a}_1\hat{a}_2^\dagger - \hat{a}_1^\dagger\hat{a}_2)$ must take the form

$$\hat{L}_z = i\hbar \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \text{ with e-values } \lambda = \pm i\hbar \text{ and e-vectors } \begin{bmatrix} 1, & 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0, & 1 \end{bmatrix}$$

$$\text{therefore } |E_{n=1}\rangle = \frac{1}{\sqrt{2}}\,|E_{x=0}, E_{y=1}\rangle + \frac{1}{\sqrt{2}}\,|E_{x=1}, E_{y=0}\rangle$$

10.18

a) **Determine the ground state energy**

This problem can be solved similarly to a one dimensional particle in a box because there is no singularity as found in the infinite spherical well. To do so, we must find solutions to the radial equation at the lowest allowed angular momentum, $l = 0$, inside the well, $a < r < b$

$$E\,R(r) = \left[-\frac{\hbar^2}{2\mu}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right) + \frac{l(l+1)\hbar^2}{2\mu r^2} + V(r)\right] R(r)$$

We will do the following substitution to make finding a solution easier

$$R(r) = \frac{u(r)}{r}$$

and thus we can simplify, also using $l = 0$ and $V(a < r < b) = 0$

$$E\,u(r) = \left(-\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{l(l+1)\hbar^2}{2\mu r^2} + V(r) \right) u(r)$$

$$\frac{d^2 u(r)}{dr^2} + k^2 u(r) = 0$$

with $k = \sqrt{\frac{2\mu E}{\hbar^2}}$, which satisfy the boundary conditions

$$u(r = a) = u(r = b) = 0 \tag{14}$$

We can make the Ansatz, assuming no time dependence

$$\begin{aligned} u(r) &= [A\sin(k(r-a)) + B\cos(k(r-a))]\mathrm{e}^{-i\omega t} && (15) \\ &= A\sin(k(r-a)) + B\cos(k(r-a)) && (16) \end{aligned}$$

where $k = \frac{n\pi}{b-a}$, the allowed wavevectors in a space of size $a < r < b$.
Given $k = \frac{n\pi}{b-a} = \sqrt{\frac{2\mu E}{\hbar^2}}$ we find the n-th level energy

$$E_n = \left(\frac{n\pi}{b-a}\right)^2 \frac{\hbar^2}{2\mu} \tag{17}$$

which at $n = 1$ is

$$E_1 = \left(\frac{\pi}{b-a}\right)^2 \frac{\hbar^2}{2\mu}$$

b) **What is the ground state position space eigenfunction up to an overall normalization constant?**

Applying the boundary conditions in Equation 14 to Equation 16, we find $B = 0$, thus

$$R(r) = \frac{A}{r}\sin(k(r-a))$$

where $k = \frac{n\pi}{b-a} = \sqrt{\frac{2\mu E}{\hbar^2}}$.

c) **What is the energy of the first excited $l = 0$ state? Explain why it would not be so straightforward to determine the energy of the $l = 1$ states.**

The first excited state of $l = 0$ is found at $n = 2$, thus from Equation 17 we find

$$E_{2,0} = \left(\frac{2\pi}{b-a}\right)^2 \frac{\hbar^2}{2\mu}$$

The energy would be much more difficult to solve for the $l = 1$ case because the radial equation does not simplify to something we can easily guess the solution to. We would be required to use the Froebenius method, which is tedious, dumb, and explained here https://www.youtube.com/watch?v=SS6bniyB7rw.

10.19

Begin by converting to center of mass coordinates. Also use page 146 of the Townsend textbook to see that $-\frac{\hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_1}{\hbar^2} \to \frac{3}{4}$ for the ground-state case.

$$\begin{aligned}
\frac{\hat{\boldsymbol{p}}_1^2}{2m_1} + \frac{\hat{\boldsymbol{p}}_2^2}{2m_2} &= \cancelto{0}{\frac{\hat{\boldsymbol{P}}^2}{2M}} + \frac{\hat{\boldsymbol{p}}^2}{2\mu} \\
\mu &= \frac{m_1 m_2}{m_1 + m_2} \\
\hat{H} &= \frac{\hat{\boldsymbol{p}}^2}{2\mu} + V_a(|\hat{\boldsymbol{r}}|) + \left(\frac{1}{4} - \frac{\hat{\boldsymbol{S}}_1 \cdot \hat{\boldsymbol{S}}_1}{\hbar^2}\right) V_b(|\hat{\boldsymbol{r}}|) \\
Eu(r) &= \left[-\frac{\hbar^2}{2\mu}\frac{d^2\mu}{dr^2} + \cancelto{0}{\frac{l(l+1)k^2}{2\mu r^2}} + V(r)\right] u(r)
\end{aligned}$$

Zone I $\quad (0 < r < b)$

$$\begin{aligned}
\hat{H} &= \frac{\hat{\boldsymbol{p}}^2}{2\mu} \\
Eu(r) &= -\frac{\hbar^2}{2\mu}\frac{d^2u}{dr^2} + V(r)u(r) \\
\frac{d^2u}{dr^2} &= -\frac{2\mu E}{\hbar^2}u(r) \\
\frac{d^2u}{dr^2} &= -k^2 u(r), \quad k = \sqrt{\frac{2\mu E}{\hbar^2}}
\end{aligned}$$

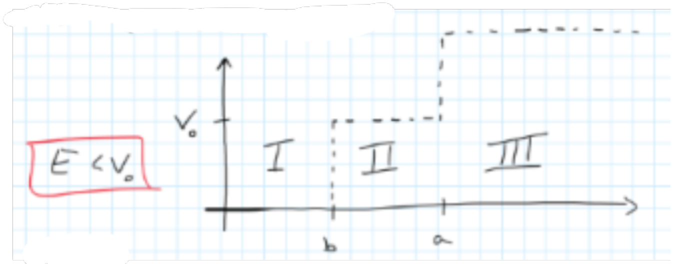

Figure 2: Representation of the potential in the problem

Zone II $\quad (b < r < a)$

$$\hat{H} = \frac{\hat{\boldsymbol{p}}^2}{2\mu} + V_0$$
$$Eu(r) = -\frac{\hbar^2}{2\mu}\frac{d^2u}{dr^2} + V_0 u(r)$$
$$\frac{d^2u}{dr^2} = \frac{2\mu}{\hbar^2}(V_0 - E)u$$
$$\frac{d^2u}{dr^2} = q_0^2 u(r), \quad q_0 = \sqrt{\frac{2\mu}{\hbar^2}(V_0 - E)}$$

Zone III $\quad (a < r < \infty)$

$$\hat{H} = \frac{\hat{\boldsymbol{p}}^2}{2\mu} + 2V_0$$
$$Eu(r) = -\frac{\hbar^2}{2\mu}\frac{d^2u}{dr^2} + V_0 u(r)$$
$$\frac{d^2u}{dr^2} = \frac{2\mu}{\hbar^2}(2V_0 - E)u$$
$$\frac{d^2u}{dr^2} = q_1^2 u(r), \quad q_0 = \sqrt{\frac{2\mu}{\hbar^2}(2V_0 - E)}$$

Now we apply boundary conditions. The wave functions and their first derivatives must be equal at the boundaries. Also, the wave function should be zero at $r = 0$ and $r = \infty$. If you are unsure as why this is, please see the discussion given in section 10.1 of the Townsend textbook.

$$u(r) = Asin(kr) + Bcos(kr) \quad (0 < r < b)$$
$$u(r) = Ce^{-qr} + De^{qr} \quad (b < r < a)$$
$$u(r) = Ee^{-qr} + Fe^{qr} \quad (a < r < \infty)$$

From the $u(r) = 0$ at $r = 0$ and r=∞ requirements it is obvious that the coefficients B and F must both be zero.