Trivial 'proof' of P vs. NP

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Theorem: $NP \not\subseteq TIME(n^{100})$.

Proof: Let $f(n) = n^{100}$ and $g(n) = n^{101}$ which are time-constructible functions. We claim $f(n) \log f(n) = o(g(n))$ since

$$\lim_{n\to\infty} \frac{f(n)\log f(n)}{g(n)} = \lim_{n\to\infty} \frac{n^{100}\log(n^{100})}{n^{101}} = \lim_{n\to\infty} \frac{100\log n}{n} = 0$$

Then by the time-hierarchy theorem, we have that

$$TIME(n^{100}) \not\subset TIME(n^{101})$$

which means there exists a language L such that $L \in TIME(n^{101})$ but $L \notin TIME(n^{100})$. Since $P = \cup_{c\in\mathbb{N}} TIME(n^c)$, by definition

$$TIME(n^{101}) \subseteq P$$

Then

$$L \in TIME(n^{101}) \subseteq P \subseteq NP$$

but $L \notin TIME(n^{100})$ so we conclude $NP \notin TIME(n^{100})$.

□

Note that the proof is not specific to the exponent being 100; the argument can be modified to show that for any constant $c \in \mathbb{N}$, we have $NP \not\subseteq TIME(n^c)$.

Question: Why does this not show that $NP \not\subseteq P$?

Explanation: For any $d \in \mathbb{N}$, $NP \not\subseteq TIME(n^d)$ is simply a consequence of the fact that $P \subseteq NP$ since

$$P = \cup_{c\in\mathbb{N}} TIME(n^c) \subseteq NP$$

and

$$P = \cup_{c\in\mathbb{N}} TIME(n^c) \not\subseteq TIME(n^d)$$

Put another way, we know $P \not\subseteq TIME(n^d)$ so $NP \not\subseteq TIME(n^d)$ whether $NP \not\subseteq P$ or $NP \subseteq P$.