Exercise 4.23

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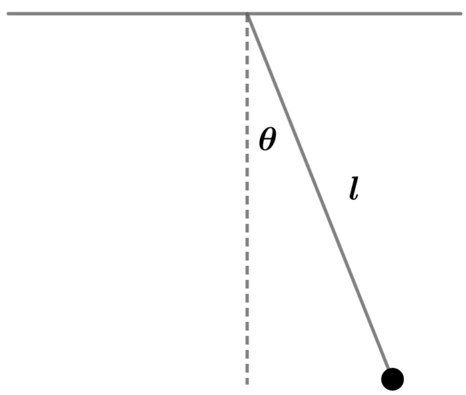

a) Applying $F = ma$ in the tangential direction gives

$$-mg\sin\theta = m(l\ddot{\theta})$$

Note that

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta}$$

so

$$-mg\sin\theta = ml\dot{\theta}\frac{d\dot{\theta}}{d\theta}$$

$$-\int_{\theta_0}^{\theta} mg\sin\theta'\,d\theta' = \int_0^{\dot{\theta}} ml\dot{\theta}'\,d\dot{\theta}'$$

Simplifying, the equation of motion for a pendulum released from rest at θ_0 is

$$mg\cos\theta - mg\cos\theta_0 = \frac{ml\dot{\theta}^2}{2}$$

Rearranging,

$$\dot{\theta} = \frac{d\theta}{dt} = \sqrt{\frac{2g\cos\theta - 2g\cos\theta_0}{l}}$$

$$dt = \sqrt{\frac{l}{2g}} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

We can integrate to find the period,

$$\int_0^{T/4} dt = \sqrt{\frac{l}{2g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}$$

Thus,

$$\boxed{T = \sqrt{\frac{8l}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}}$$

b) Using the identity

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$$

we have

$$T = \sqrt{\frac{8l}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{1 - 2\sin^2\frac{\theta}{2} - 1 + 2\sin^2\frac{\theta_0}{2}}} = 2\sqrt{\frac{l}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2})}}$$

Make the change of variables

$$\sin x \equiv \frac{\sin(\theta/2)}{\sin(\theta_0/2)}$$

$$\cos x \, dx = \frac{\cos(\theta/2)}{2\sin(\theta_0/2)} \, d\theta$$

We have

$$T = 2\sqrt{\frac{l}{g}} \int_0^{\theta_0} \frac{d\theta}{\sin\frac{\theta_0}{2}\sqrt{1 - \frac{\sin^2(\theta/2)}{\sin^2(\theta_0/2)}}} = 2\sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{\frac{2\sin(\theta_0/2)}{\cos(\theta/2)} \cos x \, dx}{\sin\frac{\theta_0}{2}\sqrt{1 - \sin^2 x}}$$

Note

$$\cos\frac{\theta}{2} = \sqrt{1 - \sin^2\frac{\theta}{2}} = \sqrt{1 - \sin^2\frac{\theta_0}{2}\sin^2 x}$$

so

$$T = 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - \sin^2\frac{\theta_0}{2}\sin^2 x}}$$

Expanding to lowest order in θ_0,

$$\sin^2\frac{\theta_0}{2} = \frac{\theta_0^2}{4} + ...$$

$$\left(1-\frac{\theta_0^2}{4}\sin^2 x\right)^{-1/2}=1+\frac{\theta_0^2}{8}\sin^2 x+...$$

$$T=4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\left(1+\frac{\theta_0^2}{8}\sin^2 x+...\right)\,dx=4\sqrt{\frac{l}{g}}\left(\frac{\pi}{2}+\frac{\theta_0^2}{8}\int_0^{\pi/2}\sin^2 x\,dx+...\right)$$

$$T=4\sqrt{\frac{l}{g}}\left(\frac{\pi}{2}+\frac{\pi\theta_0^2}{32}+...\right)$$

Hence,

$$\boxed{T=2\pi\sqrt{\frac{l}{g}}\left(1+\frac{\theta_0^2}{16}+...\right)}$$