Exercise 4.22

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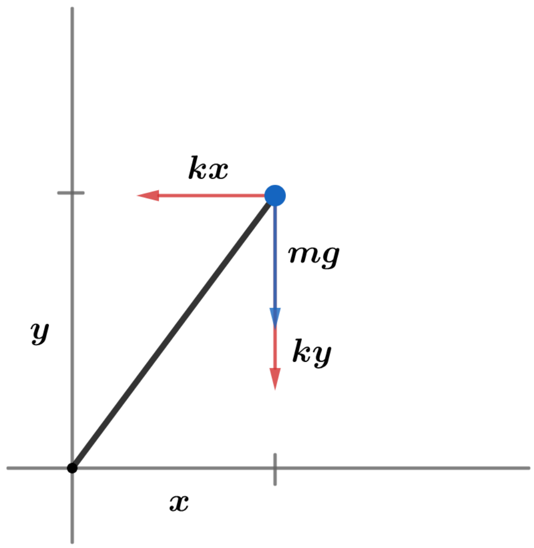

a) Suppose the mass is located at (x, y). We always have gravity and we can decompose the spring force along the axes. Applying $F = ma$,

$$m\ddot{x} = -kx$$
$$m\ddot{y} = -ky - mg$$

Defining $\omega = \sqrt{\frac{k}{m}}$,

$$\ddot{x} + \omega^2 x = 0$$
$$\ddot{y} + \omega^2 y = -g$$

The general solution for x is

$$x = A_x \sin(\omega t + \phi_x)$$

For y, we can guess $y = \alpha$ as a particular solution

$$0 + \omega^2 \alpha = -g$$
$$\alpha = -\frac{g}{\omega^2}$$

so the general solution is

$$y = y_g + y_p = A_y \sin(\omega t + \phi_y) - \frac{g}{\omega^2}$$

We are given initial conditions

$$x(0) = y(0) = 0$$
$$\dot{x}(0) = v_0 \cos\theta$$
$$\dot{y}(0) = v_0 \sin\theta$$

Thus,

$$x(0) = A_x \sin\phi_x = 0$$
$$\dot{x}(0) = \omega A_x \cos\phi_x = v_0 \cos\theta$$
$$\phi_x = 0$$
$$A_x = \frac{v_0 \cos\theta}{\omega}$$

so

$$\boxed{x(t) = \frac{v_0 \cos\theta}{\omega} \sin\omega t}$$

We also have

$$y(0) = A_y \sin\phi_y - \frac{g}{\omega^2} = 0$$
$$\dot{y}(0) = \omega A_y \cos\phi_y = v_0 \sin\theta$$

$$A_y \sin\phi_y = \frac{g}{\omega^2}$$
$$A_y \cos\phi_y = \frac{v_0 \sin\theta}{\omega}$$

$$A_y = \sqrt{\left(\frac{g}{\omega^2}\right)^2 + \left(\frac{v_0 \sin\theta}{\omega}\right)^2}$$
$$\tan\phi_y = \frac{g}{\omega v_0 \sin\theta}$$

so

$$\boxed{y(t) = \sqrt{\left(\frac{g}{\omega^2}\right)^2 + \left(\frac{v_0 \sin\theta}{\omega}\right)^2} \sin\left[\omega t + \arctan\left(\frac{g}{\omega v_0 \sin\theta}\right)\right] - \frac{g}{\omega^2}}$$

b) For small ω,

$$x(t) = \frac{v_0 \cos\theta}{\omega} \sin\omega t \approx \frac{v_0 \cos\theta}{\omega}(\omega t) = v_0 t \cos\theta$$

$$y(t) = A_y \sin(\omega t + \phi_y) - \frac{g}{\omega^2} = A_y(\sin\omega t \cos\phi_y + \cos\omega t \sin\phi_y) - \frac{g}{\omega^2}$$

Define $r = g/\omega v_0 \sin\theta$:

$$\cos\phi_y = \cos\arctan r = \frac{1}{\sqrt{r^2+1}} \approx \frac{1}{r}$$
$$\sin\phi_y = \sin\arctan r = \frac{r}{\sqrt{r^2+1}} \approx 1$$

Then

$$y(t) = A_y\left[(\sin\omega t)\left(\frac{1}{r}\right) + (\cos\omega t)\,(1)\right] - \frac{g}{\omega^2} \approx A_y\left[(\omega t)\left(\frac{1}{r}\right) + \left(1 - \frac{\omega^2 t^2}{2}\right)(1)\right] - \frac{g}{\omega^2}$$

Note that

$$A_y = \frac{v_0\sin\theta}{\omega}\sqrt{\left(\frac{g}{\omega v_0\sin\theta}\right)^2 + 1} = \frac{v_0\sin\theta}{\omega}\sqrt{r^2+1} \approx \frac{v_0\sin\theta}{\omega} r$$

Thus, combining all terms

$$y(t) = \frac{v_0\sin\theta}{\omega} r\left(\frac{\omega t}{r} + 1 - \frac{\omega^2 t^2}{2}\right) - \frac{g}{\omega^2}$$
$$y(t) = \frac{g}{\omega^2}\left(\frac{\omega^2 v_0 t\sin\theta}{g} + 1 - \frac{\omega^2 t^2}{2}\right) - \frac{g}{\omega^2}$$
$$y(t) = v_0 t\sin\theta - \frac{gt^2}{2}$$

Hence, small ω corresponds to $r \gg 1$:

$$\frac{g}{\omega v_0\sin\theta} \gg 1$$
$$\boxed{\omega \ll \frac{g}{v_0\sin\theta}}$$

For large ω,

$$x(t) = \frac{v_0\cos\theta}{\omega}\sin\omega t$$

$$y(t) = A_y\sin(\omega t + \phi_y) - \frac{g}{\omega^2} = A_y(\sin\omega t\cos\phi_y + \cos\omega t\sin\phi_y) - \frac{g}{\omega^2}$$

$$\cos\phi_y = \cos\arctan r = \frac{1}{\sqrt{r^2+1}} \approx 1$$
$$\sin\phi_y = \sin\arctan r = \frac{r}{\sqrt{r^2+1}} \approx r$$
$$A_y = \frac{v_0\sin\theta}{\omega}\sqrt{r^2+1} \approx \frac{v_0\sin\theta}{\omega}$$

Thus, combining all terms

$$y(t) \approx \frac{v_0\sin\theta}{\omega}(\sin\omega t(1) + \cos\omega t(r)) - \frac{g}{\omega^2} = \frac{v_0\sin\theta}{\omega}(\sin\omega t + r\cos\omega t - r)$$

$$y(t) \approx \frac{v_0 \sin\theta}{\omega} \sin\omega t$$

Hence, large ω corresponds to $r \ll 1$:

$$\frac{g}{\omega v_0 \sin\theta} \ll 1$$

$$\boxed{\omega \gg \frac{g}{v_0 \sin\theta}}$$

c) We want the projectile to hit the ground when it is moving straight downward:

$$y = 0$$

$$\dot{x} = 0$$

so

$$y(t) = A_y \sin(\omega t + \phi_y) - \frac{g}{\omega^2} = A_y(\sin\omega t \cos\phi_y + \cos\omega t \sin\phi_y) - \frac{g}{\omega^2} = 0$$

and

$$\dot{x}(t) = v_0 \cos\theta \cos\omega t = 0 \to \cos\omega t = 0, \sin\omega t = 1$$

Substituting these conditions into the first equation,

$$A_y((1)\cos\phi_y + (0)\sin\phi_y) - \frac{g}{\omega^2} = 0$$

Substituting A_y,

$$\left(\frac{v_0 \sin\theta}{\omega}\sqrt{r^2+1}\right)\cos\phi_y - \frac{g}{\omega^2} = 0$$

Substituting $\cos\phi_y$,

$$\left(\frac{v_0 \sin\theta}{\omega}\sqrt{r^2+1}\right)\left(\frac{1}{\sqrt{r^2+1}}\right) - \frac{g}{\omega^2} = 0$$

$$\frac{v_0 \sin\theta}{\omega} - \frac{g}{\omega^2} = 0$$

Hence,

$$\boxed{\omega = \frac{g}{v_0 \sin\theta}}$$