Exercise 3.67

Kevin S. Huang

We have a general position vector:

$$\vec{r} = x\hat{x} + y\hat{y} = r\cos\theta\,\hat{x} + r\sin\theta\,\hat{y}$$

Differentiating once,

$$\dot{\vec{r}} = (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)\,\hat{x} + (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)\,\hat{y}$$

Differentiating again,

$$\begin{aligned}\ddot{\vec{r}} = (\ddot{r}\cos\theta - \dot{r}\dot{\theta}\sin\theta - \dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta)\,\hat{x}\\ +(\ddot{r}\sin\theta + \dot{r}\dot{\theta}\cos\theta + \dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta)\,\hat{y}\end{aligned}$$

Simplifying,

$$\begin{aligned}\ddot{\vec{r}} = (\ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta)\,\hat{x}\\ +(\ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta)\,\hat{y}\end{aligned}$$

Recall

$$\begin{aligned}\hat{r} &= \cos\theta\,\hat{x} + \sin\theta\,\hat{y}\\ \hat{\theta} &= -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}\end{aligned}$$

Thus,

$$\begin{aligned}\ddot{\vec{r}} &= (\ddot{r} - r\dot{\theta}^2)(\cos\theta\,\hat{x} + \sin\theta\,\hat{y}) + (r\ddot{\theta} + 2\dot{r}\dot{\theta})(-\sin\theta\,\hat{x} + \cos\theta\,\hat{y})\\ \ddot{\vec{r}} &= (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}\end{aligned}$$

Because $\vec{F} = F_r\hat{r} + F_\theta\hat{\theta} = m\ddot{\vec{r}}$,

$$\begin{aligned}F_r &= m(\ddot{r} - r\dot{\theta}^2)\\ F_\theta &= m(r\ddot{\theta} + 2\dot{r}\dot{\theta})\end{aligned}$$