Transfer Matrix

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$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi(x) = E\psi(x)$$

$$\psi(x) = Ae^{ikx} + Be^{-ikx}$$

$$k = \sqrt{\frac{2m(E-V)}{\hbar^2}}$$

Potential Function

$$V(x) =$$

$$\begin{array}{cc} V_0 & x < d_0 \\ V_1 & d_0 < x < d_1 \\ V_2 & d_1 < x < d_2 \\ ... & \\ V_{n-1} & d_{n-2} < x < d_{n-1} \\ V_n & x > d_{n-1} \end{array}$$

Time-Independent Schrodinger Equation for Arbitrary V_m

$$0 \leq m \leq n-1$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_m}{dx^2} + V_m\psi_m = E\psi_m$$

$$\frac{d^2\psi_m}{dx^2} = -{k_m}^2\psi_m$$

$$k_m = \sqrt{\frac{2m(E-V_m)}{\hbar^2}}$$

$$\psi_m = C_{m1}e^{ik_mx} + C_{m2}e^{-ik_mx}$$

$$\psi_{m+1} = C_{(m+1)1}e^{ik_{m+1}x} + C_{(m+1)2}e^{-ik_{m+1}x}$$

Boundary Conditions

$$\psi_m(d_m) = \psi_{m+1}(d_m)$$

$$\psi'_m(d_m) = \psi'_{m+1}(d_m)$$

$$C_{m1}e^{ik_md_m} + C_{m2}e^{-ik_md_m} = C_{(m+1)1}e^{ik_{m+1}d_m} + C_{(m+1)2}e^{-ik_{m+1}d_m}$$

$$k_m(C_{m1}e^{ik_md_m} - C_{m2}e^{-ik_md_m}) = k_{m+1}(C_{(m+1)1}e^{ik_{m+1}d_m} - C_{(m+1)2}e^{-ik_{m+1}d_m})$$

Transfer Matrix

Theory:

$$\begin{pmatrix} F \\ G \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}$$

Application:

$$\begin{pmatrix} C_{(m+1)1} \\ C_{(m+1)2} \end{pmatrix} = M_m \begin{pmatrix} C_{m1} \\ C_{m2} \end{pmatrix}$$

$$\begin{pmatrix} C_{n1} \\ C_{n2} \end{pmatrix} = M_{n-1}M_{n-2}...M_2M_1M_0 \begin{pmatrix} C_{01} \\ C_{02} \end{pmatrix}$$

Scattering from left
C_{01} - incoming amplitude from left
C_{02} - reflected amplitude
C_{n1} - transmitted amplitude
$C_{n2} = 0$ - incoming amplitude from right

$$C_{(m+1)1}e^{ik_{m+1}d_m} + C_{(m+1)2}e^{-ik_{m+1}d_m} = C_{m1}e^{ik_md_m} + C_{m2}e^{-ik_md_m}$$

$$C_{(m+1)1}e^{ik_{m+1}d_m} - C_{(m+1)2}e^{-ik_{m+1}d_m} = \frac{k_m}{k_{m+1}}(C_{m1}e^{ik_md_m} - C_{m2}e^{-ik_md_m})$$

$$2C_{(m+1)1}e^{ik_{m+1}d_m} = \left(1 + \frac{k_m}{k_{m+1}}\right)C_{m1}e^{ik_md_m} + \left(1 - \frac{k_m}{k_{m+1}}\right)C_{m2}e^{-ik_md_m}$$

$$C_{(m+1)1} = \frac{1}{2}\left(1 + \frac{k_m}{k_{m+1}}\right)e^{-i(k_{m+1}-k_m)d_m}C_{m1} + \frac{1}{2}\left(1 - \frac{k_m}{k_{m+1}}\right)e^{-i(k_{m+1}+k_m)d_m}C_{m2}$$

$$2C_{(m+1)2}e^{-ik_{m+1}d_m} = \left(1 - \frac{k_m}{k_{m+1}}\right)C_{m1}e^{ik_md_m} + \left(1 + \frac{k_m}{k_{m+1}}\right)C_{m2}e^{-ik_md_m}$$

$$C_{(m+1)2} = \frac{1}{2}\left(1 - \frac{k_m}{k_{m+1}}\right)e^{i(k_{m+1}+k_m)d_m}C_{m1} + \frac{1}{2}\left(1 + \frac{k_m}{k_{m+1}}\right)e^{i(k_{m+1}-k_m)d_m}C_{m2}$$

$$M_m = \begin{pmatrix} \frac{1}{2}\left(1 + \frac{k_m}{k_{m+1}}\right)e^{-i(k_{m+1}-k_m)d_m} & \frac{1}{2}\left(1 - \frac{k_m}{k_{m+1}}\right)e^{-i(k_{m+1}+k_m)d_m} \\ \frac{1}{2}\left(1 - \frac{k_m}{k_{m+1}}\right)e^{i(k_{m+1}+k_m)d_m} & \frac{1}{2}\left(1 + \frac{k_m}{k_{m+1}}\right)e^{i(k_{m+1}-k_m)d_m} \end{pmatrix}$$

$$\begin{pmatrix} C_{(m+1)1} \\ C_{(m+1)2} \end{pmatrix} = \begin{pmatrix} \frac{k_{m+1}+k_m}{2k_{m+1}}e^{-i(k_{m+1}-k_m)d_m} & \frac{k_{m+1}-k_m}{2k_{m+1}}e^{-i(k_{m+1}+k_m)d_m} \\ \frac{k_{m+1}-k_m}{2k_{m+1}}e^{i(k_{m+1}+k_m)d_m} & \frac{k_{m+1}+k_m}{2k_{m+1}}e^{i(k_{m+1}-k_m)d_m} \end{pmatrix} \begin{pmatrix} C_{m1} \\ C_{m2} \end{pmatrix}$$

Formulas

$$\begin{pmatrix} C_{n1} \\ C_{n2} \end{pmatrix} = M_{n-1}M_{n-2}...M_2M_1M_0 \begin{pmatrix} C_{01} \\ C_{02} \end{pmatrix}$$

$$M_m = \begin{pmatrix} \frac{k_{m+1}+k_m}{2k_{m+1}}e^{-i(k_{m+1}-k_m)d_m} & \frac{k_{m+1}-k_m}{2k_{m+1}}e^{-i(k_{m+1}+k_m)d_m} \\ \frac{k_{m+1}-k_m}{2k_{m+1}}e^{i(k_{m+1}+k_m)d_m} & \frac{k_{m+1}+k_m}{2k_{m+1}}e^{i(k_{m+1}-k_m)d_m} \end{pmatrix}$$

Transmission and Reflection
Scattering from the left:

$$F = M_{11}A + M_{12}B$$

$$0 = M_{21}A + M_{22}B$$

$$F = \frac{M_{11}M_{22} - M_{12}M_{21}}{M_{22}}A$$

$$F = \frac{|M|}{M_{22}}A$$

$$B = -\frac{M_{21}}{M_{22}}A$$

$$T = \frac{|F|^2}{|A|^2} = \frac{|M|^2}{|M_{22}|^2}$$

$$R = \frac{|B|^2}{|A|^2} = \frac{|M_{21}|^2}{|M_{22}|^2}$$