Dropping a pebble into a well

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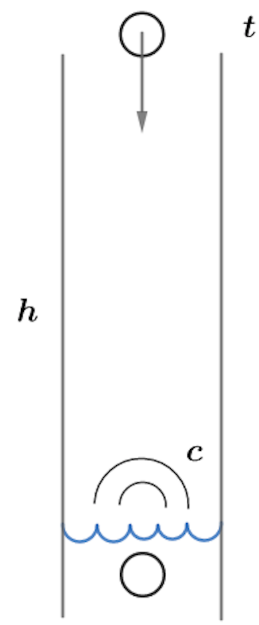
Problem: A pebble is dropped into a well and the splash is heard time t later. Assuming the speed of sound is c, what is the depth h of the well?



Solution: The time t includes both the time t_1 it takes the pebble to hit the water and the time t_2 it takes sound to travel back up. Thus,

$$t = t_1 + t_2$$

From kinematics, we have

$$h = \frac{1}{2} g t_1^2$$
$$h = c t_2$$

Substituting back,

$$t = \sqrt{\frac{2h}{g}} + \frac{h}{c}$$

Simplifying,

$$\left(t - \frac{h}{c}\right)^2 = \frac{2h}{g}$$

$$\frac{h^2}{c^2} - 2\left(\frac{t}{c} + \frac{1}{g}\right)h + t^2 = 0$$

$$h = \frac{2\left(\frac{t}{c} + \frac{1}{g}\right) \pm \sqrt{4\left(\frac{t}{c} + \frac{1}{g}\right)^2 - \frac{4t^2}{c^2}}}{\frac{2}{c^2}} = \frac{c^2}{2}\left[2\left(\frac{t}{c} + \frac{1}{g}\right) \pm \sqrt{4\left(\frac{2t}{cg} + \frac{1}{g^2}\right)}\right]$$

Clearly, if $t = 0$ then $h = 0$ so we take the negative sign:

$$h = c^2\left[\left(\frac{t}{c} + \frac{1}{g}\right) - \sqrt{\frac{2t}{cg} + \frac{1}{g^2}}\right] = \frac{c^2}{g}\left(\frac{c + gt}{c} - \sqrt{\frac{c + 2gt}{c}}\right)$$

Therefore,

$$\boxed{h = \frac{c^2}{g}\left(1 + \frac{gt}{c} - \sqrt{1 + \frac{2gt}{c}}\right)}$$

As a check, consider the case $c \to \infty$. We can expand

$$\sqrt{1 + x} = 1 + \frac{x}{2} - \frac{x^2}{8} + ...$$

Thus,

$$h \approx \frac{c^2}{g}\left[1 + \frac{gt}{c} - \left(1 + \frac{gt}{c} - \frac{g^2t^2}{2c^2}\right)\right] = \frac{1}{2}gt^2$$

which we expected.