Exercise 3.52

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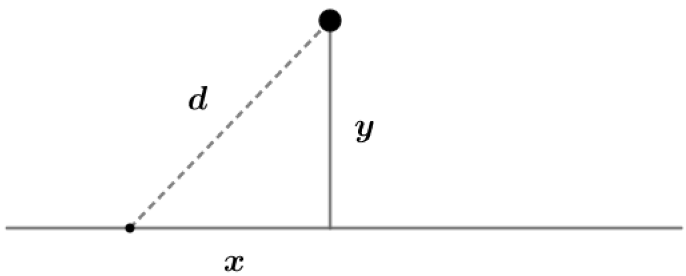


a) The distance the ball is from the starting point is given by

$$d^2 = x^2 + y^2$$

The rate at which this increases is

$$\frac{d}{dt}(d^2) = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2(xv_x + yv_y)$$

From kinematics, we have

$$x(t) = (v\cos\theta)t$$
$$y(t) = (v\sin\theta)t - \frac{1}{2}gt^2$$
$$v_x(t) = v\cos\theta$$
$$v_y(t) = v\sin\theta - gt$$

We want to find the maximum angle at which the distance never decreases during the flight.

$$\frac{d}{dt}(d^2) = 2(xv_x + yv_y)$$
$$= [(v\cos\theta)t][v\cos\theta] + [(v\sin\theta)t - \frac{1}{2}gt^2][v\sin\theta - gt]$$
$$= v^2\cos^2\theta + (v\sin\theta - \frac{1}{2}gt)(v\sin\theta - gt)$$
$$= v^2\cos^2\theta + v^2\sin^2\theta - \frac{3}{2}gt(v\sin\theta) + \frac{1}{2}g^2t^2$$

Let's minimize the expression in t,

$$\frac{d}{dt}\left[-\frac{3}{2}gt(v\sin\theta) + \frac{1}{2}g^2t^2\right] = 0$$

$$-\frac{3}{2}gv\sin\theta + g^2t = 0$$

$$t_c = \frac{3v\sin\theta}{2g} = \frac{3}{4}T$$

Substituting back,

$$\frac{d}{dt}(d^2)_{t_c} = v^2 - \frac{3}{2}g\left(\frac{3v\sin\theta}{2g}\right)v\sin\theta + \frac{1}{2}g^2\left(\frac{3v\sin\theta}{2g}\right)^2$$

$$= v^2 - \frac{9v^2\sin^2\theta}{4} + \frac{9v^2\sin^2\theta}{8} = \left(1 - \frac{9}{8}\sin^2\theta\right)v^2$$

Since we require $\frac{d}{dt}(d^2)_{t_c} \geq 0$,

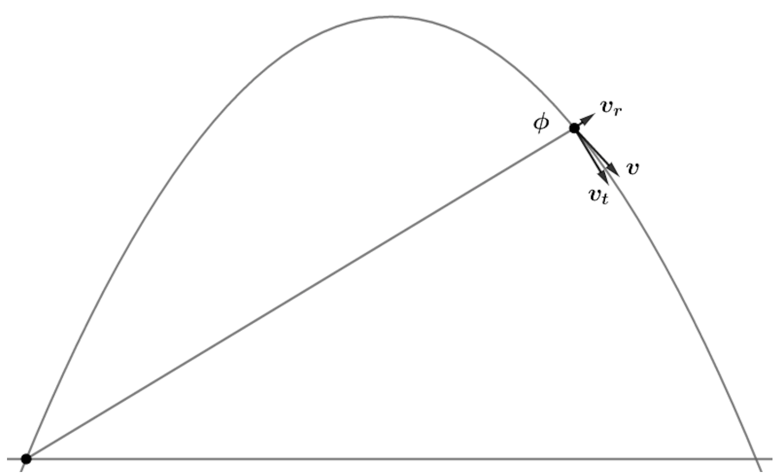
$$1 - \frac{9}{8}\sin^2\theta \geq 0$$

$$\sin^2\theta \leq \frac{8}{9}$$

$$\sin\theta \leq \frac{2\sqrt{2}}{3}$$

$$\theta_{max} = \arcsin\frac{2\sqrt{2}}{3} \approx 70.5^\circ$$

b) This maximum angle equals the minimum angle from Exercise 3.51. Let's project the ball's velocity onto a radial component away from the starting point and a tangential component. At t_c, the rate of change of d is minimized so ϕ (angle between v_r and v) is maximized.



In this problem, we can increase our initial angle until $\phi(t_c) = \frac{\pi}{2}$ since that corresponds to $v_r = 0 \rightarrow \frac{d}{dt}(d) = 0$. Exercise 3.51 is equivalent (we reverse the launch direction) to asking for the smallest initial angle such that the ball lands perpendicular to the plane. Thus, we can decrease our launch angle until $\phi(t_c) = \frac{\pi}{2}$ since $\phi(t_c)$ is the maximum angle between the ball's velocity and the plane. Therefore, these two angles are the same.