Solutions - Chapter 3

Kevin S. Huang

Problem 3.1

a)

$$[\hat{A}, \hat{B} + \hat{C}] = \hat{A}(\hat{B} + \hat{C}) - (\hat{B} + \hat{C})\hat{A} = \hat{A}\hat{B} - \hat{B}\hat{A} + \hat{A}\hat{C} - \hat{C}\hat{A} = [\hat{A}, \hat{B}] + [\hat{A}, \hat{C}]$$

b)

$$i[\hat{A}, \hat{B}\hat{C}] = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{C}\hat{A} = \hat{A}\hat{B}\hat{C} - \hat{B}\hat{A}\hat{C} + \hat{B}\hat{A}\hat{C} - \hat{B}\hat{C}\hat{A} = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C}$$

Problem 3.2

Eigenstate problem:

$$\hat{S}_n |\mu\rangle = \mu\frac{\hbar}{2} |\mu\rangle$$

$$\hat{S}_n = S_x \sin\theta\cos\phi + S_y \sin\theta\sin\phi + S_z \cos\theta$$

$$= \frac{\hbar}{2}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\sin\theta\cos\phi + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\sin\theta\sin\phi + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\cos\theta\right]$$

$$= \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta(\cos\phi - i\sin\phi) \\ \sin\theta(\cos\phi + i\sin\phi) & -\cos\theta \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}$$

Matrix form:

$$\frac{\hbar}{2}\begin{pmatrix} \cos\theta & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} \langle +\mathbf{z}|\mu\rangle \\ \langle -\mathbf{z}|\mu\rangle \end{pmatrix} = \mu\frac{\hbar}{2}\begin{pmatrix} \langle +\mathbf{z}|\mu\rangle \\ \langle -\mathbf{z}|\mu\rangle \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta - \mu & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta - \mu \end{pmatrix}\begin{pmatrix} \langle +\mathbf{z}|\mu\rangle \\ \langle -\mathbf{z}|\mu\rangle \end{pmatrix} = 0$$

$$\begin{vmatrix} \cos\theta - \mu & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta - \mu \end{vmatrix} = \mu^2 - \cos^2\theta - \sin^2\theta = 0$$

Eigenvalues:

$$\mu = \pm 1$$

$$\begin{pmatrix} \cos\theta \mp 1 & e^{-i\phi}\sin\theta \\ e^{i\phi}\sin\theta & -\cos\theta \mp 1 \end{pmatrix} \begin{pmatrix} \langle +\mathbf{z} | \pm \mathbf{n} \rangle \\ \langle -\mathbf{z} | \pm \mathbf{n} \rangle \end{pmatrix} = 0$$

Equation:

$$(\cos\theta \mp 1) \langle +\mathbf{z} | \pm \mathbf{n} \rangle + e^{-i\phi}\sin\theta \langle -\mathbf{z} | \pm \mathbf{n} \rangle = 0$$

Normalization:

$$|\langle +\mathbf{z} | \pm \mathbf{n} \rangle|^2 + |\langle -\mathbf{z} | \pm \mathbf{n} \rangle|^2 = 1$$

$$|\langle +\mathbf{z} | \pm \mathbf{n} \rangle|^2 + \frac{(\cos\theta \mp 1)^2}{\sin^2\theta} |\langle +\mathbf{z} | \pm \mathbf{n} \rangle|^2 = 1$$

$$|\langle +\mathbf{z} | \pm \mathbf{n} \rangle|^2 = \frac{\sin^2\theta}{\sin^2\theta + (\cos\theta \mp 1)^2} = \frac{\sin^2\theta}{2 \mp 2\cos\theta} = \frac{(1-\cos\theta)(1+\cos\theta)}{2(1 \mp \cos\theta)}$$

$$|\langle +\mathbf{z} | + \mathbf{n} \rangle| = \sqrt{\frac{1+\cos\theta}{2}} = \cos\frac{\theta}{2}$$

$$|\langle +\mathbf{z} | - \mathbf{n} \rangle| = \sqrt{\frac{1-\cos\theta}{2}} = \sin\frac{\theta}{2}$$

$$|\langle -\mathbf{z} | + \mathbf{n} \rangle| = e^{i\phi}\sin\frac{\theta}{2}$$

$$|\langle -\mathbf{z} | - \mathbf{n} \rangle| = -e^{i\phi}\cos\frac{\theta}{2}$$

Eigenstates:

$$|+\mathbf{n}\rangle = \cos\frac{\theta}{2}|+\mathbf{z}\rangle + e^{i\phi}\sin\frac{\theta}{2}|-\mathbf{z}\rangle$$

$$|-\mathbf{n}\rangle = \sin\frac{\theta}{2}|+\mathbf{z}\rangle - e^{i\phi}\cos\frac{\theta}{2}|-\mathbf{z}\rangle$$

Problem 3.3

$$\sigma_1^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_2^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_3^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_1\sigma_2 + \sigma_2\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 0$$

$$\sigma_1\sigma_3 + \sigma_3\sigma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0$$

$$\sigma_2\sigma_3 + \sigma_3\sigma_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = 0$$

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij} I$$

Problem 3.4

a)

$$\boldsymbol{\sigma} \times \boldsymbol{\sigma} = (\sigma_x \boldsymbol{i} + \sigma_y \boldsymbol{j} + \sigma_z \boldsymbol{k}) \times (\sigma_x \boldsymbol{i} + \sigma_y \boldsymbol{j} + \sigma_z \boldsymbol{k})$$

$$= (\sigma_y\sigma_z - \sigma_z\sigma_y)\boldsymbol{i} + (\sigma_z\sigma_x - \sigma_x\sigma_z)\boldsymbol{j} + (\sigma_x\sigma_y - \sigma_y\sigma_x)\boldsymbol{k}$$

$$[\sigma_y, \sigma_z] = 2i\sigma_x$$
$$[\sigma_z, \sigma_x] = 2i\sigma_y$$
$$[\sigma_x, \sigma_y] = 2i\sigma_z$$

$$\boldsymbol{\sigma} \times \boldsymbol{\sigma} = 2i\boldsymbol{\sigma}$$

b)

$$\boldsymbol{\sigma} \cdot \boldsymbol{a} \boldsymbol{\sigma} \cdot \boldsymbol{b} = (a_x\sigma_x + a_y\sigma_y + a_z\sigma_z)(b_x\sigma_x + b_y\sigma_y + b_z\sigma_z)$$

$$= (a_xb_x + a_yb_y + a_zb_z)I + a_xb_y\sigma_x\sigma_y + a_xb_z\sigma_x\sigma_z + a_yb_x\sigma_y\sigma_x + a_yb_z\sigma_y\sigma_z + a_zb_x\sigma_z\sigma_x + a_zb_y\sigma_z\sigma_y$$

$$= \boldsymbol{a} \cdot \boldsymbol{b} I + i\sigma_x(a_yb_z - a_zb_y) - i\sigma_y(a_xb_z - a_zb_x) + i\sigma_z(a_xb_y - a_yb_x)$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{a} \boldsymbol{\sigma} \cdot \boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b} I + i\boldsymbol{\sigma} \cdot (\boldsymbol{a} \times \boldsymbol{b})$$

Problem 3.5

a)

$$\hat{R}(\theta\boldsymbol{j}) = e^{-i\hat{S}_y\theta/\hbar} = 1 - \left(\frac{i\theta}{\hbar}\right)\hat{S}_y + \frac{1}{2!}\left(\frac{i\theta}{\hbar}\right)^2 \hat{S}_y{}^2 - \frac{1}{3!}\left(\frac{i\theta}{\hbar}\right)^3 \hat{S}_y{}^3 + ...$$

$$\hat{S}_y{}^2 = I$$

$$\hat{R}(\theta\boldsymbol{j}) = 1 - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2 + \frac{1}{4!}\left(\frac{\theta}{2}\right)^4 + ...$$

$$+ i\sigma_y \left[\left(\frac{\theta}{2}\right) - \frac{1}{3!}\left(\frac{\theta}{2}\right)^3 + ...\right]$$

$$\hat{R}(\theta \boldsymbol{j}) = \cos\frac{\theta}{2} + \frac{2i}{\hbar}\hat{S}_y \sin\frac{\theta}{2}$$

b)

$$\hat{R}(\theta \boldsymbol{j}) \underset{S_z}{\rightarrow} \begin{pmatrix} \cos(\theta/2) & 0 \\ 0 & \cos(\theta/2) \end{pmatrix} + \begin{pmatrix} 0 & \sin(\theta/2) \\ -\sin(\theta/2) & 0 \end{pmatrix}$$

$$\hat{R}(\theta \boldsymbol{j}) \left|+\mathbf{z}\right\rangle = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$\phi = 0$:

$$\left|+\mathbf{n}\right\rangle = \cos\frac{\theta}{2}\left|+\mathbf{z}\right\rangle + \sin\frac{\theta}{2}\left|-\mathbf{z}\right\rangle$$

Problem 3.6

$$\hat{J}_- \left|j, m\right\rangle = c_-\hbar \left|j, m-1\right\rangle$$

$$\hat{J}_+\hat{J}_- = (\hat{J}_x + i\hat{J}_y)(\hat{J}_x - i\hat{J}_y) = \hat{J}_x^{\;2} + \hat{J}_y^{\;2} - i[\hat{J}_x, \hat{J}_y] = \hat{\boldsymbol{J}}^2 - \hat{J}_z^{\;2} + \hbar\hat{J}_z$$

$$\left\langle j, m\right| \hat{J}_+\hat{J}_- \left|j, m\right\rangle = c_-^* c_- \hbar^2 \left\langle j, m-1 | j, m-1\right\rangle$$

$$\left\langle j, m\right| \hat{\boldsymbol{J}}^2 - \hat{J}_z^{\;2} + \hbar\hat{J}_z \left|j, m\right\rangle = [j(j+1) - m^2 + m]\hbar^2 \left\langle j, m | j, m\right\rangle$$

$$c_- = \sqrt{j(j+1) - m(m-1)}\left|j, m-1\right\rangle$$

Problem 3.7

$$(\left\langle\alpha\right| + \lambda^* \left\langle\beta\right|)(\left|\alpha\right\rangle + \lambda\left|\beta\right\rangle) \geq 0$$

$$\left\langle\alpha|\alpha\right\rangle + \lambda^*\left\langle\beta|\alpha\right\rangle + \lambda\left\langle\alpha|\beta\right\rangle + \lambda^*\lambda\left\langle\beta|\beta\right\rangle \geq 0$$

Choose:

$$\lambda = -\frac{\left\langle\beta|\alpha\right\rangle}{\left\langle\beta|\beta\right\rangle}$$

$$\left\langle\alpha|\alpha\right\rangle - \frac{\left\langle\alpha|\beta\right\rangle\left\langle\beta|\alpha\right\rangle}{\left\langle\beta|\beta\right\rangle} - \frac{\left\langle\beta|\alpha\right\rangle\left\langle\alpha|\beta\right\rangle}{\left\langle\beta|\beta\right\rangle} + \frac{\left\langle\alpha|\beta\right\rangle\left\langle\beta|\alpha\right\rangle}{\left\langle\beta|\beta\right\rangle} = 0$$

$$\left\langle\alpha|\alpha\right\rangle \geq \frac{\left\langle\beta|\alpha\right\rangle\left\langle\alpha|\beta\right\rangle}{\left\langle\beta|\beta\right\rangle}$$

Schwarz inequality:

$$\left\langle\alpha|\alpha\right\rangle\left\langle\beta|\beta\right\rangle \geq |\left\langle\alpha|\beta\right\rangle|^2$$

Problem 3.8

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} = i\hat{C}$$

Hermitian:

$$\langle\alpha|A|\beta\rangle^* = \langle\beta|A|\alpha\rangle$$

$$\langle\alpha|AB - BA|\beta\rangle^* = \langle\alpha|AB|\beta\rangle^* - \langle\alpha|BA|\beta\rangle^* = \langle\beta|BA|\alpha\rangle - \langle\beta|AB|\alpha\rangle = \langle\beta|BA - AB|\alpha\rangle$$

$$BA - AB = -iC$$

$$\langle\alpha|iC|\beta\rangle^* = -i\,\langle\alpha|C|\beta\rangle = \langle\alpha| - iC|\beta\rangle$$

C is Hermitian:

$$\langle\alpha|C|\beta\rangle^* = \langle\beta|C|\alpha\rangle$$

Problem 3.9

Eigenstate of $\hat{S}_z$: $|+\mathbf{z}\rangle$

$$\Delta S_x = \sqrt{\langle {S_x}^2\rangle - \langle S_x\rangle^2} = \frac{\hbar}{2}$$

$$\langle S_x\rangle = \langle +\mathbf{z}|S_x| + \mathbf{z}\rangle = 0$$

$$\langle {S_x}^2\rangle = \langle +\mathbf{z}|{S_x}^2| + \mathbf{z}\rangle = \left(\frac{\hbar}{2}\right)^2$$

$$\Delta S_y = \sqrt{\langle {S_y}^2\rangle - \langle S_y\rangle^2} = \frac{\hbar}{2}$$

$$\langle S_y\rangle = \langle +\mathbf{z}|S_y| + \mathbf{z}\rangle = 0$$

$$\langle {S_y}^2\rangle = \langle +\mathbf{z}|{S_y}^2| + \mathbf{z}\rangle = \left(\frac{\hbar}{2}\right)^2$$

$$\langle S_z\rangle = \langle +\mathbf{z}|S_z| + \mathbf{z}\rangle = \frac{\hbar}{2}$$

$$\Delta S_x \Delta S_y \geq \frac{\hbar|\langle S_z\rangle|}{2}$$

$$\left(\frac{\hbar}{2}\right)\left(\frac{\hbar}{2}\right) \geq \left(\frac{\hbar}{2}\right)\left(\frac{\hbar}{2}\right)$$

Eigenstate of $\hat{S}_x$: $|+\mathbf{x}\rangle$

$$\Delta S_x = \sqrt{\langle {S_x}^2\rangle - \langle S_x\rangle^2} = 0$$

$$\langle S_x \rangle = \langle +\mathbf{x}|S_x| + \mathbf{x}\rangle = \frac{\hbar}{2}$$

$$\langle {S_x}^2 \rangle = \langle +\mathbf{x}|{S_x}^2| + \mathbf{x}\rangle = \left(\frac{\hbar}{2}\right)^2$$

$$\Delta S_y = \sqrt{\langle {S_y}^2\rangle - \langle S_y\rangle^2} = \frac{\hbar}{2}$$

$$\langle S_y \rangle = \langle +\mathbf{x}|S_y| + \mathbf{x}\rangle = 0$$

$$\langle {S_y}^2 \rangle = \langle +\mathbf{x}|{S_y}^2| + \mathbf{x}\rangle = \left(\frac{\hbar}{2}\right)^2$$

$$\langle S_z \rangle = \langle +\mathbf{x}|S_z| + \mathbf{x}\rangle = 0$$

$$\Delta S_x \Delta S_y \geq \frac{\hbar|\langle S_z\rangle|}{2}$$

$$0 \geq 0$$

Problem 3.10

$$S_x \underset{S_z}{\rightarrow} \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y \underset{S_z}{\rightarrow} \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$S_x \underset{S_z}{\rightarrow} \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z$$

$$S_xS_y - S_yS_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \left(\frac{\hbar}{2}\right)^2\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \left(\frac{\hbar}{2}\right)^2\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \frac{i\hbar^2}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\hbar\hat{S}_z$$

Problem 3.11

Instead of inserting the identity element connecting the two bases, we can use a cyclic substitution $x \rightarrow y$, $y \rightarrow z$, $z \rightarrow x$:

$$\hat{S}_y \underset{S_y}{\rightarrow} \left(\frac{\hbar}{2}\right)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{S}_z \underset{S_y}{\longrightarrow} \left(\frac{\hbar}{2}\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{S}_x \underset{S_y}{\longrightarrow} \left(\frac{\hbar}{2}\right) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Problem 3.12

$$\hat{S}_z \left|+\mathbf{z}\right\rangle = \frac{\hbar}{2} \left|+\mathbf{z}\right\rangle$$

$$\hat{S}_z \left|+\mathbf{z}\right\rangle = -\frac{\hbar}{2} \left|-\mathbf{z}\right\rangle$$

$$\hat{S}_z = \frac{\hbar}{2} \left|+\mathbf{z}\right\rangle \left\langle+\mathbf{z}\right| - \frac{\hbar}{2} \left|-\mathbf{z}\right\rangle \left\langle-\mathbf{z}\right|$$

$$\hat{S_+} \left|+\mathbf{z}\right\rangle = 0$$

$$\hat{S_+} \left|-\mathbf{z}\right\rangle = \hbar \left|+\mathbf{z}\right\rangle$$

$$\hat{S_-} \left|+\mathbf{z}\right\rangle = \hbar \left|-\mathbf{z}\right\rangle$$

$$\hat{S_-} \left|-\mathbf{z}\right\rangle = 0$$

$$\hat{S_+} = \hbar \left|+\mathbf{z}\right\rangle \left\langle-\mathbf{z}\right|$$

$$\hat{S_-} = \hbar \left|-\mathbf{z}\right\rangle \left\langle+\mathbf{z}\right|$$

Problem 3.13

$$\hat{J}_x \underset{J_z}{\longrightarrow} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\hat{J}_y \underset{J_z}{\longrightarrow} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\hat{J}_z \underset{J_z}{\longrightarrow} \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$[\hat{J}_x, \hat{J}_y] = \hat{J}_x \hat{J}_y - \hat{J}_y \hat{J}_x$$

$$\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} - \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= i\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} = i\hbar \hat{J}_z$$

Problem 3.14

Angular Momentum (spin-1: $j = 1$):

$$\hat{J}_z |j, m\rangle = m\hbar |j, m\rangle$$

$$\hat{J}_z |1, 1\rangle = \hbar |1, 1\rangle$$
$$\hat{J}_z |1, 0\rangle = 0$$
$$\hat{J}_z |1, -1\rangle = -\hbar |1, 1\rangle$$

$$\hat{J}_z = \hbar |1, 1\rangle \langle 1, 1| - \hbar |1, -1\rangle \langle 1, -1| \underset{J_z}{\rightarrow} \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\hat{J}_+ |1, 1\rangle = 0$$
$$\hat{J}_+ |1, 0\rangle = \sqrt{2}\hbar |1, 1\rangle$$
$$\hat{J}_+ |1, -1\rangle = \sqrt{2}\hbar |1, 0\rangle$$

$$\hat{J}_+ = \sqrt{2}\hbar(|1, 1\rangle \langle 1, 0| + |1, 0\rangle \langle 1, -1|) \underset{J_z}{\rightarrow} \sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\hat{J}_- |1, 1\rangle = \sqrt{2}\hbar |1, 0\rangle$$
$$\hat{J}_- |1, 0\rangle = \sqrt{2}\hbar |1, -1\rangle$$
$$\hat{J}_- |1, -1\rangle = 0$$

$$\hat{J}_- = \sqrt{2}\hbar(|1, 0\rangle \langle 1, 1| + |1, -1\rangle \langle 1, 0|) \underset{J_z}{\rightarrow} \sqrt{2}\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\hat{J}_x = \frac{J_+ + J_-}{2}$$
$$\hat{J}_y = \frac{J_+ - J_-}{2i}$$

$$\hat{J}_x \underset{J_z}{\rightarrow} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\hat{J}_y \underset{J_z}{\rightarrow} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

Problem 3.15

$$\hat{S}_x \left|1, \mu\right\rangle_x = \mu\hbar \left|1, \mu\right\rangle_x$$

$$\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mu\hbar \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

Eigenvalues: $\mu = 1, 0, -1$

$$\begin{pmatrix} -\mu & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\mu & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\mu \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$\mu = 1$:

$$-a + \frac{1}{\sqrt{2}} b = 0$$

$$\frac{1}{\sqrt{2}} b - c = 0$$

$$b = \sqrt{2} a$$

$$c = a$$

Normalization:

$$a^2 + 2a^2 + a^2 = 1$$

$$a = \frac{1}{2}$$

$$\left|1, 1\right\rangle_x = \frac{1}{2} \left|1, 1\right\rangle + \frac{\sqrt{2}}{2} \left|1, 0\right\rangle + \frac{1}{2} \left|1, -1\right\rangle$$

$\mu = 0$:

$$\frac{1}{\sqrt{2}} b = 0$$

$$\frac{1}{\sqrt{2}} a + \frac{1}{\sqrt{2}} c = 0$$

$$b = 0$$

$$c = -a$$

Normalization:

$$a^2 + 0 + a^2 = 1$$

$$a = \frac{\sqrt{2}}{2}$$

$$|1,0\rangle_x = \frac{\sqrt{2}}{2}|1,1\rangle - \frac{\sqrt{2}}{2}|1,-1\rangle$$

$\mu = -1$:

$$a + \frac{1}{\sqrt{2}}b = 0$$

$$\frac{1}{\sqrt{2}}b + c = 0$$

$$b = -\sqrt{2}a$$

$$c = a$$

Normalization:

$$a^2 + 2a^2 + a^2 = 1$$

$$a = \frac{1}{2}$$

$$|1,-1\rangle_x = \frac{1}{2}|1,1\rangle - \frac{\sqrt{2}}{2}|1,0\rangle + \frac{1}{2}|1,-1\rangle$$

Problem 3.16

$$P = |\,\langle 1,0|_x|1,1\rangle\,|^2 = \frac{1}{2}$$

$$\langle 1,0|_x = \frac{\sqrt{2}}{2}\langle 1,1| - \frac{\sqrt{2}}{2}\langle 1,-1|$$

$$\langle 1,0|_x|1,1\rangle = \frac{\sqrt{2}}{2}\langle 1,1|1,1\rangle - \frac{\sqrt{2}}{2}\langle 1,-1|1,1\rangle = \frac{\sqrt{2}}{2}$$

Problem 3.17

$$|\psi\rangle \underset{S_z}{\rightarrow} \frac{1}{\sqrt{14}}\begin{pmatrix} 1 \\ 2 \\ 3i \end{pmatrix}$$

a)

$$P(S_z = \hbar) = |\,\langle 1,1|\psi\rangle\,|^2 = \frac{1}{14}$$

$$P(S_z = 0) = |\,\langle 1,0|\psi\rangle\,|^2 = \frac{2}{7}$$

$$P(S_z = -\hbar) = |\,\langle 1,-1|\psi\rangle\,|^2 = \frac{9}{14}$$

$$\langle S_z\rangle = P(S_z = \hbar)(\hbar) + P(S_z = 0)(0) + P(S_z = -\hbar)(-\hbar) = -\frac{4}{7}\hbar$$

b)

$$\langle S_x \rangle = \langle \psi | S_x | \psi \rangle$$

$$= \frac{1}{\sqrt{14}} \begin{pmatrix} 1 & 2 & -3i \end{pmatrix} \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \frac{1}{\sqrt{14}} \begin{pmatrix} 1 \\ 2 \\ 3i \end{pmatrix} = \frac{2}{7\sqrt{2}}\hbar$$

c)

$$P(S_x = \hbar) = | \langle 1, 1_x | \psi \rangle |^2$$

$$\langle 1, 1|_x = \frac{1}{2} \langle 1, 1| + \frac{\sqrt{2}}{2} \langle 1, 0| + \frac{1}{2} \langle 1, -1|$$

$$|\psi\rangle = \frac{1}{\sqrt{14}} |1, 1\rangle + \frac{2}{\sqrt{14}} |1, 0\rangle + \frac{3i}{\sqrt{14}} |1, -1\rangle$$

$$\langle 1, 1_x | \psi \rangle = \frac{1}{2\sqrt{14}} + \frac{2\sqrt{2}}{2\sqrt{14}} + \frac{3i}{2\sqrt{14}} = \frac{1 + 2\sqrt{2} + 3i}{2\sqrt{14}}$$

$$P(S_x = \hbar) = | \langle 1, 1_x | \psi \rangle |^2 = \left(\frac{1 + 2\sqrt{2}}{2\sqrt{14}} \right)^2 + \left(\frac{3}{2\sqrt{14}} \right)^2 = \frac{1 + 4\sqrt{2} + 8 + 9}{4(14)} = \frac{9 + 2\sqrt{2}}{28}$$

Problem 3.18

$$\hat{S}_n |\mu\rangle = \mu\hbar |\mu\rangle$$

$$\hat{S}_n = S_x \sin\theta\cos\phi + S_y \sin\theta\sin\phi + S_z \cos\theta$$

$$= \left[\frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \sin\theta\cos\phi + \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \sin\theta\sin\phi + \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \cos\theta \right]$$

$$= \hbar \begin{pmatrix} \cos\theta & \frac{\sin\theta\cos\phi - i\sin\theta\sin\phi}{\sqrt{2}} & 0 \\ \frac{\sin\theta\cos\phi + i\sin\theta\sin\phi}{\sqrt{2}} & 0 & \frac{\sin\theta\cos\phi - i\sin\theta\sin\phi}{\sqrt{2}} \\ 0 & \frac{\sin\theta\cos\phi + i\sin\theta\sin\phi}{\sqrt{2}} & -\cos\theta \end{pmatrix}$$

$$= \hbar \begin{pmatrix} \cos\theta & \frac{e^{-i\phi}\sin\theta}{\sqrt{2}} & 0 \\ \frac{e^{i\phi}\sin\theta}{\sqrt{2}} & 0 & \frac{e^{-i\phi}\sin\theta}{\sqrt{2}} \\ 0 & \frac{e^{i\phi}\sin\theta}{\sqrt{2}} & -\cos\theta \end{pmatrix}$$

$$\hbar \begin{pmatrix} \cos\theta & \frac{e^{-i\phi}\sin\theta}{\sqrt{2}} & 0 \\ \frac{e^{i\phi}\sin\theta}{\sqrt{2}} & 0 & \frac{e^{-i\phi}\sin\theta}{\sqrt{2}} \\ 0 & \frac{e^{i\phi}\sin\theta}{\sqrt{2}} & -\cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mu\hbar \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta-\mu & \frac{e^{-i\phi}\sin\theta}{\sqrt{2}} & 0 \\ \frac{e^{i\phi}\sin\theta}{\sqrt{2}} & -\mu & \frac{e^{-i\phi}\sin\theta}{\sqrt{2}} \\ 0 & \frac{e^{i\phi}\sin\theta}{\sqrt{2}} & -\cos\theta-\mu \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

Eigenvalues:

$$(\cos\theta-\mu)\left[(-\mu)(-\cos\theta-\mu)-\frac{e^{-i\phi}\sin\theta}{\sqrt{2}}\frac{e^{i\phi}\sin\theta}{\sqrt{2}}\right]-\frac{e^{-i\phi}\sin\theta}{\sqrt{2}}\left[\frac{e^{i\phi}\sin\theta}{\sqrt{2}}(-\cos\theta-\mu)\right]=0$$

$$(\cos\theta-\mu)\left[(-\mu)(-\cos\theta-\mu)-\frac{\sin^2\theta}{2}\right]-\frac{\sin^2\theta}{2}(-\cos\theta-\mu)=0$$

$$(-\mu)(\mu^2-\cos^2\theta)-\frac{\sin^2\theta}{2}(\cos\theta-\mu)+\frac{\sin^2\theta}{2}(\cos\theta+\mu)=0$$

$$(-\mu)(\mu^2-\cos^2\theta)-\frac{\sin^2\theta}{2}(\cos\theta-\mu)+\frac{\sin^2\theta}{2}(\cos\theta+\mu)=0$$

$$(-\mu)(\mu^2-\cos^2\theta-\sin^2\theta)=0$$

$$\mu=1,0,-1$$

$\mu=1$:

$$(\cos\theta-1)a+\frac{e^{-i\phi}\sin\theta}{\sqrt{2}}b=0$$

$$\frac{e^{i\phi}\sin\theta}{\sqrt{2}}b-(\cos\theta+1)c=0$$

$$c=\frac{1-\cos\theta}{1+\cos\theta}a$$

$$b=\frac{e^{i\phi}(1-\cos\theta)\sqrt{2}}{\sin\theta}a$$

Normalization:

$$a^2+\frac{2(1-\cos\theta)^2}{\sin^2\theta}a^2+\frac{(1-\cos\theta)^2}{(1+\cos\theta)^2}a^2=1$$

$$a=\frac{1+\cos\theta}{2}=\cos^2\left(\frac{\theta}{2}\right)$$

$$|1,1\rangle_n=\cos^2\left(\frac{\theta}{2}\right)|1,1\rangle+\frac{e^{i\phi}\sin\theta\sqrt{2}}{2}|1,0\rangle+\sin^2\left(\frac{\theta}{2}\right)|1,-1\rangle$$

$\mu = 0$:

$$(\cos\theta)a + \frac{e^{-i\phi}\sin\theta}{\sqrt{2}}b = 0$$

$$\frac{e^{i\phi}\sin\theta}{\sqrt{2}}b - (\cos\theta)c = 0$$

$$b = -\frac{e^{i\phi}\cos\theta\sqrt{2}}{\sin\theta}a$$
$$c = -a$$

Normalization:

$$a^2 + \frac{2\cos^2\theta}{\sin^2\theta}a^2 + a^2 = 1$$

$$a = \frac{\sin\theta\sqrt{2}}{2}$$

$$|1,0\rangle_n = \frac{\sin\theta\sqrt{2}}{2}|1,1\rangle - e^{i\phi}\cos\theta\,|1,0\rangle - \frac{\sin\theta\sqrt{2}}{2}|1,-1\rangle$$

$\mu = -1$:

$$(\cos\theta + 1)a + \frac{e^{-i\phi}\sin\theta}{\sqrt{2}}b = 0$$

$$\frac{e^{i\phi}\sin\theta}{\sqrt{2}}b - (\cos\theta - 1)c = 0$$

$$c = \frac{1+\cos\theta}{1-\cos\theta}a$$

$$b = -\frac{e^{i\phi}(1+\cos\theta)\sqrt{2}}{\sin\theta}a$$

Normalization:

$$a^2 + \frac{2(1+\cos\theta)^2}{\sin^2\theta}a^2 + \frac{(1+\cos\theta)^2}{(1-\cos\theta)^2}a^2 = 1$$

$$a = \frac{1-\cos\theta}{2} = \sin^2\left(\frac{\theta}{2}\right)$$

$$|1,-1\rangle_n = \sin^2\left(\frac{\theta}{2}\right)|1,1\rangle - \frac{e^{i\phi}\sin\theta\sqrt{2}}{2}|1,0\rangle + \cos^2\left(\frac{\theta}{2}\right)|1,-1\rangle$$

Problem 3.19

$$\hat{R}(\theta\boldsymbol{j}) = e^{-i\hat{S}_y\theta/\hbar} = 1 + \left(-\frac{i\theta}{\hbar}\right)\hat{S}_y + \frac{1}{2!}\left(-\frac{i\theta}{\hbar}\right)^2\hat{S}_y{}^2 + \frac{1}{3!}\left(-\frac{i\theta}{\hbar}\right)^3\hat{S}_y{}^3 + ...$$

$$\hat{S}_y = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

$$\hat{S}_y{}^2 = \left(\frac{\hbar}{\sqrt{2}}\right)^2 \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\hat{S}_y{}^3 = \left(\frac{\hbar}{\sqrt{2}}\right)^3 \begin{pmatrix} 0 & -2i & 0 \\ 2i & 0 & -2i \\ 0 & 2i & 0 \end{pmatrix} = \hbar^2 \hat{S}_y$$

$$\hat{S}_y{}^4 = \left(\frac{\hbar}{\sqrt{2}}\right)^4 \begin{pmatrix} 2 & 0 & -2 \\ 0 & 4 & 0 \\ -2 & 0 & 2 \end{pmatrix} = \hbar^2 \hat{S}_y{}^2$$

$$\hat{R}(\theta \boldsymbol{j}) = 1 + \frac{1}{2!}\left(-\frac{i\theta}{\hbar}\right)^2 \hat{S}_y{}^2 + \frac{1}{4!}\left(-\frac{i\theta}{\hbar}\right)^4 \hat{S}_y{}^4 + ... + \left(-\frac{i\theta}{\hbar}\right)\hat{S}_y + \frac{1}{3!}\left(-\frac{i\theta}{\hbar}\right)^3 \hat{S}_y{}^3$$

$$= 1 + \frac{S_y{}^2}{2}\left[-\frac{1}{2!}(\theta)^2 + \frac{1}{4!}(\theta)^4 - ...\right] + ... + \frac{-iS_y}{\sqrt{2}}\left[(\theta) - \frac{1}{3!}(\theta)^3 + ...\right]$$

$$= 1 + \frac{S_y{}^2}{2}(\cos\theta - 1) - \frac{iS_y}{\sqrt{2}}\sin\theta$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \frac{\cos\theta - 1}{2} & 0 & \frac{1-\cos\theta}{2} \\ 0 & \cos\theta - 1 & 0 \\ \frac{1-\cos\theta}{2} & 0 & \frac{1-\cos\theta}{2} \end{pmatrix} + \begin{pmatrix} 0 & -\frac{\sin\theta}{\sqrt{2}} & 0 \\ \frac{\sin\theta}{\sqrt{2}} & 0 & -\frac{\sin\theta}{\sqrt{2}} \\ 0 & \frac{\sin\theta}{\sqrt{2}} & 0 \end{pmatrix}$$

$$\hat{R}(\theta \boldsymbol{j}) = \begin{pmatrix} \frac{1+\cos\theta}{2} & -\frac{\sin\theta}{\sqrt{2}} & \frac{1-\cos\theta}{2} \\ \frac{\sin\theta}{\sqrt{2}} & \cos\theta & -\frac{\sin\theta}{\sqrt{2}} \\ \frac{1-\cos\theta}{2} & \frac{\sin\theta}{\sqrt{2}} & \frac{1+\cos\theta}{2} \end{pmatrix}$$

$$\hat{R}(\theta \boldsymbol{j})\,|1,1\rangle = \cos^2\left(\frac{\theta}{2}\right)|1,1\rangle + \frac{\sin\theta\sqrt{2}}{2} + \sin^2\left(\frac{\theta}{2}\right) = |1,1\rangle_n \quad (\phi = 0)$$

Problem 3.20

a)

$$f = |\langle S_n = \hbar | S_z = \hbar\rangle|^2 |\langle S_z = -\hbar | S_n = \hbar\rangle|^2 = |\langle 1,1|_n |1,1\rangle|^2 |\langle 1,-1||1,1\rangle_n|^2$$

$$|1,1\rangle_n = \cos^2\left(\frac{\theta}{2}\right)|1,1\rangle + \frac{\sin\theta\sqrt{2}}{2}|1,0\rangle + \sin^2\left(\frac{\theta}{2}\right)|1,-1\rangle$$

$$\langle 1,1|_n = \cos^2\left(\frac{\theta}{2}\right)\langle 1,1| + \frac{\sin\theta\sqrt{2}}{2}\langle 1,0| + \sin^2\left(\frac{\theta}{2}\right)\langle 1,-1|$$

$$\langle 1,1|_n|1,1\rangle = \cos^2\left(\frac{\theta}{2}\right)$$

$$\langle 1,-1||1,1\rangle_n = \sin^2\left(\frac{\theta}{2}\right)$$

$$f = \sin^4\left(\frac{\theta}{2}\right)\cos^4\left(\frac{\theta}{2}\right) = \frac{\sin^4\theta}{16}$$

b)

$$\theta = \pm\frac{\pi}{2}$$

$$f_{max} = \frac{1}{16}$$

c)

$$f = |\langle 1,-1|1,1\rangle|^2 = 0$$

Last $SG\mathbf{z}$ device transmits particles with $S_z = 0$ only:

a)

$$f = |\langle 1,1|_n|1,1\rangle|^2|\langle 1,0||1,1\rangle_n|^2$$

$$\langle 1,1|_n|1,1\rangle = \cos^2\left(\frac{\theta}{2}\right)$$

$$\langle 1,0||1,1\rangle_n = \frac{\sin\theta\sqrt{2}}{2}$$

$$f = \cos^4\left(\frac{\theta}{2}\right)\frac{\sin^2\theta}{2} = \frac{\sin^2\theta(1+\cos\theta)^2}{8}$$

b)

$$\frac{d}{d\theta}\left[\sin^2\theta(1+\cos\theta)^2\right] = 2\sin\theta\cos\theta(1+\cos\theta)^2 + 2\sin^2\theta(1+\cos\theta)(-\sin\theta) = 0$$

$\sin\theta = 0$
$1+\cos\theta = 0$:

$$\cos\theta(1+\cos\theta) - \sin^2\theta = 2\cos^2\theta + \cos\theta - 1 = 0$$

$$(2\cos\theta - 1)(\cos\theta + 1) = 0$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \pm\frac{\pi}{3}$$

$$f_{max} = \frac{27}{128}$$

c)

$$f = |\langle 1,0|1,1\rangle|^2 = 0$$

Problem 3.21

$$J_z = m\hbar$$

$$\mathbf{J} = \sqrt{j(j+1)}\hbar$$

$$\cos\theta = \frac{m}{\sqrt{j(j+1)}}$$

a) spin-$\frac{1}{2}$

$$\cos\theta = \frac{1/2}{\sqrt{(1/2)(1/2+1)}} = \frac{\sqrt{3}}{3}$$

$$\theta = \arccos\frac{\sqrt{3}}{3}$$

b) spin-1

$$\cos\theta = \frac{1}{\sqrt{(1)(1+1)}} = \frac{\sqrt{2}}{2}$$

$$\theta = \frac{\pi}{4}$$

c) macroscopic spinning top

$$L_z = \mathbf{L}$$

$$\theta = 0$$

Problem 3.22

$$|\psi\rangle = \sum_{a,b} c_{a,b}\,|a,b\rangle$$

$$\hat{A}\hat{B}\,|\psi\rangle = \sum_{a,b} c_{a,b}\hat{A}\hat{B}\,|a,b\rangle = \sum_{a,b} c_{a,b}\hat{A}b\,|a,b\rangle = \sum_{a,b} c_{a,b}ab\,|a,b\rangle$$

$$\hat{B}\hat{A}\,|\psi\rangle = \sum_{a,b} c_{a,b}\hat{B}\hat{A}\,|a,b\rangle = \sum_{a,b} c_{a,b}\hat{B}a\,|a,b\rangle = \sum_{a,b} c_{a,b}ba\,|a,b\rangle$$

$$\hat{A}\hat{B} = \hat{B}\hat{A}$$

$$[\hat{A},\hat{B}] = 0$$