Relativity

Relativity

- Seen as an intricate theory that is necessary when dealing with really high speeds
- Two charged initially stationary particles: Electrostatic force
- In another, moving reference frame, there is also a magnetic force
- How could force depend on the choice of the inertial frame?
- Electromagnetism needs relativity for an explanation

Applications

- Photons (much of optics) are always relativistic
- GPS measuring the time for a radio signal to travel from satellites
- Particle physics
- Quantum field theory: relativistic
- General relativity – relativistic theory of gravity
- Astrophysics and cosmology

Postulates of Special Relativity

1. The laws of physics are the same in every inertial reference frame.

Inertial frame: objects which no force acts upon move in a straight line with constant velocity

2. The speed of light in vacuum (c) is the same in every inertial reference frame.

c = 299,792,458 m/s

Time Dilation

Problem 1. Consider a "light clock" that works as follows. A photon is emitted towards a mirror at a known distance l and reflected back. It is detected (almost) at the emitter again. The time from the emission to the detection (a "tick") is measured to be t. Now we look at the clock from a reference frame where the whole apparatus is moving with velocity v *perpendicularly* to the light beam. Assume that the lengths perpendicular to the motion do not change. How long is the tick for us? (Hint: the light beam follows a zig-zag path.)

Fact 1: Lorentz factor

If the time interval between events happening at a stationary point is ***t***, then in a reference frame where the speed of the point is ***v*** the time interval is ***γt***.

$$\beta = v/c$$

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Length Contraction

Problem 2. Now consider the same "light clock" as in Problem 1., but moving *in parallel* to the light beam, with velocity v. What is the distance to the mirror in our reference frame, if the distance in the stationary frame is l?

Fact 2: Length contraction

- If the length of a stationary rod is l, then its length in a reference frame moving in parallel to the rod with speed v is l/γ.

- Lengths are contracted (compressed) in the direction of motion.

Proper Time

Problem 3. A spaceship flies freely from (t_1, x_1, y_1, z_1) (event 1) to (t_2, x_2, y_2, z_2) (event 2). What is the *proper time* τ – time measured by a passenger on the spaceship – between these events?

$$c^2\tau^2 = c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2$$

Lorentz transformations

- Galilean relativity, lengths and time intervals are absolute
- Einsteinian relativity, neither is absolute
- Proper time must be independent of our reference frame

Fact 3: The spacetime interval

$$s=\sqrt{c^2(\Delta t)^2-(\Delta x)^2-(\Delta y)^2-(\Delta z)^2}$$

is independent of the choice of reference frame.

Fact 4

- If s is a real number, the interval is called time-like
- If s is imaginary, the interval is space-like
- If s is zero, the interval is light-like

The interval between two events on the same light-ray (in vacuum) is zero – thus, light-like.

Minkowski spacetime

Spacetime points are represented by position four-vectors.

$$x^{\mu} = (ct, x, y, z)$$

Introduce complex numbers to use Pythagorean theorem.

$$is = \sqrt{(ic\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$$

Fact 5

Changes of inertial reference frames correspond to rotations and shifts in the space coordinates ***ict***, ***x***, ***y***, and ***z***.

- General: Poincaré transformations
- Only rotate and do not shift: Lorentz transformations

Problem 4. Take two coordinate systems, O and O', with the spatial axes parallel and the (spatial) origin* of O' moving in the x-direction with velocity v. Calculate the angle α between the x- and x'-axis. (Hint: make a diagram with ict on one axis and x on another. Add the ict'- and x'-axes. Calculate the x- and ict-coordinate of one arbitrary point the spatial origin of O' passes through. The ratio of these coordinates is $\tan\alpha$.)

Fact 6: Lorentz boost

A Lorentz boost in the x-direction from standstill to velocity c corresponds to rotation of ***x***- and ***ict***-axis by an angle:

$$\alpha = \arctan \frac{v}{ic} = \arctan \frac{\beta}{i}$$

Problem 5. Calculate $\cos\alpha$ and $\sin\alpha$.

Fact 7: Rapidity

$$\cos\alpha = \gamma\ , \ \sin\alpha = \beta\gamma / i$$

The quantity $\varphi = \alpha / i$ is a real dimensionless number and is called the *rapidity*.

Hyperbolic Geometry

- Eliminate imaginary units and some minuses

$$\sinh \alpha = \frac{e^{\alpha} - e^{-\alpha}}{2}$$

$$\cosh \alpha = \frac{e^{\alpha} + e^{-\alpha}}{2}$$

$$\tanh \alpha = \frac{\sinh \alpha}{\cosh \alpha}$$

$$\cosh^2 \alpha - \sinh^2 \alpha = 1$$

Problem 6. Prove for the rapidity φ that $\tanh\varphi = \beta$, $\cosh\varphi = \gamma$ and $\sinh\varphi = \beta\gamma$.

Problem 7. Prove again the length contraction formula of Fact 2. Here use rotation of Minkowski spacetime.

Problem 8. Prove similarly the time dilation formula of Fact 1.

Fact 8: Velocity Addition Formula

If an object moves with velocity ***u*** with respect to frame ***O'*** and ***O'*** moves with velocity ***v*** with respect to frame ***O***, then the velocity of the object in ***O*** is:

$$w = \frac{u+v}{1+\frac{uv}{c^2}}$$

Problem 9. Prove the velocity addition formula in the last fact. (Hint: $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

and

$$\tanh(\alpha+\beta)=\frac{\tanh\alpha+\tanh\beta}{1+\tanh\alpha\tanh\beta}.)$$

Problem 10. Show that the velocity addition formula implies the postulate that the speed of light is universal. (Hint: $u = \pm c$.)

Problem 11. Prove that if u and v in the velocity addition formula are both between $-c$ and c, then so is w. (Hint: show that $\frac{dw}{du} > 0$ – hence w is monotonous – and use the result of the last problem that $u = \pm c$ corresponds to $w = \pm c$.)

Fact 9

If there exists a reference frame where an object moves slower than light, then it does so in every reference frame.

Light-Cones

- The trajectory of a particle in space-time is called its **world-line**
- The world-line of a photon cuts a special wedge in the diagram
- The inside of the wedge can be influenced by an event at the tip of the cone; the outside cannot

Fact 10

If the spacetime diagram is scaled so that ***i*** meters (on the ***ict***-axis) is at the same distance from the origin as 1 meter (on the ***x***-axis), then the world-line of a photon is at 45° from either axis.

Fact 11

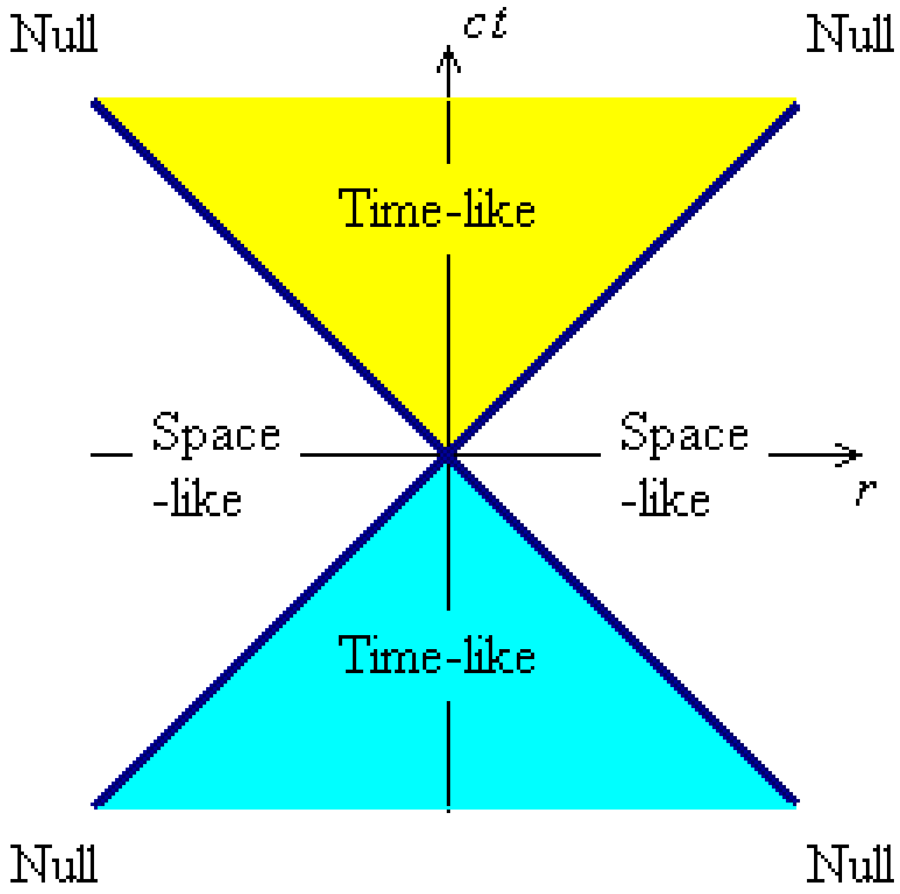
Simultaneity is relative.

Problem 12. In reference frame O, two events take place at the same time $t = 0$, but with spatial separation Δx. What is the time $\Delta t'$ between them in reference frame O', which is moving in the x-direction with velocity v?

[Answer: $\Delta t' = -\gamma v \Delta x / c^2$]

Fact 12

The order of two events with time-like or light-like separation is absolute. For space-like separation, the order depends on the reference frame.



Fact 13

Time-like separation or light-like separation allows one event to be the cause of another

Causality should hold: No information may be sent to the past

Information cannot propagate faster than light in a vacuum.

Fact 14: Lorentz transformations

When going to a reference frame moving in the ***x***-direction with velocity ***v***, the time and space coordinates of an event transform as follows:

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$x' = \gamma\left(x - vt\right)$$

$$y' = y$$

$$z' = z$$

Problem 13. Prove the last fact.

Problem 14. Show algebraically that if boosting in both x- and y-directions, the order of boosts matters.

Dynamics: Four Vector

- Collection of four numbers that transforms under Lorentz transformations

$$q^{\mu} = \left(q^{t}, q^{x}, q^{y}, q^{z}\right)$$

- Spatial components rotate just like a usual vector

$$\left(q^{x}, q^{y}, q^{z}\right) \equiv \vec{q} \qquad \left(iq^{t}, \vec{q}\right)$$

- Time and space components are mixed by Lorentz boosts that act as rotations in the four-space of

Dynamics: Four Vector

Boost in the x-direction:

$$q^t{}' = \gamma\left(q^t - \beta q^x\right)$$
$$q^x{}' = \gamma\left(q^x - \beta q^t\right)$$

Lorentz-invariant length of the four-vector:

$$|q^\mu| = \sqrt{(q^t)^2 - (q^x)^2 - (q^y)^2 - (q^z)^2}$$

Fact 15

Four-velocity:

$$v^{\mu} = \frac{d\,x^{\mu}}{d\,\tau}$$

Four-acceleration:

$$a^{\mu} = \frac{d\,v^{\mu}}{d\,\tau}$$

Problem 15. Show that the four-velocity of a particle moving with speed v in the x-direction is $(\gamma c, \gamma v, 0, 0)$.

Problem 16. What Lorentz-invariant quantity is the length of the four-velocity from the last problem?

Fact 16

The length of any four-velocity is c.

Problem 17. Show that the four-acceleration of a particle moving and accelerating in the x-direction with a three-acceleration of magnitude $a = dv/dt$ is $(\beta\gamma^4 a, \gamma^4 a, 0, 0)$ with invariant length a.

Fact 17

The four-momentum of a particle with mass ***m*** is the four vector:

$$p^{\mu} = m v^{\mu}$$

Fact 18

$$p^{\mu} = \left(E/c , \vec{p} \right)$$

Total energy:

$$E = \gamma m c^2$$

Relativistic momentum:

$$\vec{p} = \gamma m \vec{v}$$

Fact 19

The length of the four-momentum is ***mc***, regardless of what the velocity is.

$$E^2 = (pc)^2 + (mc^2)^2$$

For massless particles (such as photons), ***E = pc***.

Fact 20

In interactions, four-momentum is conserved.

Encompasses both the conservation of energy and the conservation of momentum.

Fact 21

The total energy can be separated into the rest energy and the kinetic energy.

$$E_{\text{rest}} = m c^2$$

$$E_k = (\gamma - 1) m c^2$$

Problem 18. Show that for low speeds, $E_k \approx \dfrac{m v^2}{2}$.

Energy

If an object has any internal energy, then it must be taken into account in its rest energy and its rest mass.

For any ultrarelativistic object moving with almost the speed of c, the rest energy and rest mass can be neglected, $E \approx pc$.

Speed of light, c, corresponds to $\gamma = \infty$.

Fact 22

It takes infinite energy to accelerate a massive object to c.
Massless particles move only with a speed of c.

Force

Fact 23. $$\vec{F} = \frac{d\,\vec{p}}{dt} = \frac{d(\gamma m v)}{dt}\ .$$

Fact 24. *Four-force*

$$F^{\mu} = m\,a^{\mu} = \frac{d\,p^{\mu}}{d\tau}\ .$$

Problem 19. Show that if all the motion is in x-direction, then $F^{\mu} = (\beta\gamma F, \gamma F, 0, 0)$.

In general, $F^{\mu} = (\gamma \vec{v} \cdot \vec{F}/c, \gamma \vec{F})$ where $\vec{v} \cdot \vec{F} = dE/dt$ is the *power*.

Optical effects

Problem 20. What is the apparent length of a rod with rest length l moving with velocity v in parallel to the rod, if you take into account the finite travel times of photons from its ends to our eyes?

Positive v is moving away from observer

$$L'' = \sqrt{\frac{1-\beta}{1+\beta}}\, L$$

Fact 25. Doppler shift of the frequency of light: $\nu' = \nu_0 \sqrt{\dfrac{c-v}{c+v}}$.

Problem 21. Prove the formula, considering the world-lines of two wave-crests.

Fact 26: Lorentz Force

Electromagnetic field: $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$

Separate the field into components parallel and perpendicular to $\vec{v}$:

Lorentz transformations:

$$\vec{E}_{\parallel}' = \vec{E}_{\parallel},\ \vec{B}_{\parallel}' = \vec{B}_{\parallel},$$

$$\vec{E}_{\perp}' = \gamma(\vec{E}_{\perp} + \vec{v} \times \vec{B}_{\perp}),$$

$$\vec{B}_{\perp}' = \gamma(\vec{B}_{\perp} - \vec{v} \times \vec{E}_{\perp}/c^2)$$