Gravitation

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Question

Two identical masses m are a distance l apart. The masses interact only via gravitation. Starting from rest, how long does it take before they collide? Assume $l \gg r$ where r is the radius of the masses.

Initial Energy

$$U_0 = -\frac{Gm^2}{l}$$

Conservation of Energy

r is the distance between the two masses.

$$\frac{1}{2}mv^2(r) - \frac{Gm^2}{r} = -\frac{Gm^2}{l}$$

$$v(r) = -\sqrt{2Gm\left(\frac{1}{r} - \frac{1}{l}\right)}$$

$$v = \frac{dr}{dt}$$

$$\frac{dr}{dt} = -\sqrt{2Gm\left(\frac{1}{r} - \frac{1}{l}\right)}$$

$$\int_l^0 \sqrt{\frac{lr}{l-r}}\, dr = -\int_0^T \sqrt{2Gm}\, dt$$

Integration Table:

$$\int \sqrt{\frac{r}{l-r}}\, dr = -\sqrt{r(l-r)} - l \arctan \frac{\sqrt{r(l-r)}}{r-l}$$

$$\int_l^0 \sqrt{\frac{r}{l-r}}\, dr = \sqrt{l(l-l)} + l \arctan \frac{\sqrt{r(l-l)}}{l-l}$$

We have problems so let's take the limit of an increment going to zero.

$$\int_{l-\delta}^{0} \sqrt{\frac{r}{l-r}}\, dr = \sqrt{\delta(l-\delta)} + l \arctan \frac{\sqrt{\delta(l-\delta)}}{-\delta}$$

$$\lim_{\delta \to 0} \sqrt{\delta(l-\delta)} + l \arctan \frac{\sqrt{\delta(l-\delta)}}{-\delta} = -l \lim_{\delta \to 0} \arctan \frac{\sqrt{\delta(l-\delta)}}{\delta}$$

$$-l \lim_{\delta \to 0} \arctan \frac{\sqrt{\delta(l-\delta)}}{\delta} = -l \lim_{\delta \to 0} \arctan \sqrt{\frac{l-\delta}{\delta}} = -\frac{l\pi}{2}$$

$$-\frac{l\sqrt{l}\pi}{2} = -\sqrt{2Gm}T$$

$$T = \frac{\pi l \sqrt{l}}{2\sqrt{2Gm}}$$

$$T = \frac{\pi l}{2}\sqrt{\frac{l}{2Gm}}$$

2013 F=ma exam

16. A very large number of small particles forms a spherical cloud. Initially they are at rest, have uniform mass density per unit volume p_0, and occupy a region of radius r_0. The cloud collapses due to gravitation; the particles do not interact with each other in any other way. How much time passes until the cloud collapses full?

Let's take an infinitesimal mass as the outer surface of the cloud. From conservation of energy we have:

$$\frac{1}{2}\delta m v^2 - \frac{GM\delta m}{r} = -\frac{GM\delta m}{r_0}$$

Hence:

$$\frac{1}{2}v^2 - \frac{GM}{r} = -\frac{GM}{r_0}$$

This is the same equation as we have derived above so we can use the same equation for T.

$$T = \sqrt{\frac{3\pi}{32G\rho_0}}$$