Solutions - Chapter 8

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Problem 8.1

$$\langle x', t'|x_0, t_0\rangle = \sqrt{\frac{m}{2\pi\hbar i(t'-t_0)}} e^{im(x'-x_0)^2/2\hbar(t'-t_0)}$$

$$\langle x'|\psi(t')\rangle = \int_{-\infty}^{\infty} dx_0 \ \langle x', t'|x_0, t_0\rangle \langle x_0|\psi(t_0)\rangle$$

Gaussian position-space wave packet:

$$\langle x_0|\psi(t_0)\rangle = \frac{1}{\sqrt{\sqrt{\pi}a}} e^{-x_0^2/2a^2}$$

$$\langle x'|\psi(t')\rangle = \int_{-\infty}^{\infty} dx_0 \sqrt{\frac{m}{2\pi\hbar i(t'-t_0)}} e^{im(x'-x_0)^2/2\hbar(t'-t_0)} \frac{1}{\sqrt{\sqrt{\pi}a}} e^{-x_0^2/2a^2}$$

$$= \sqrt{\frac{m}{2\pi\hbar i(t'-t_0)}} \frac{1}{\sqrt{\sqrt{\pi}a}} e^{imx'^2/2\hbar(t'-t_0)} \int_{-\infty}^{\infty} dx_0 \, e^{im(-2x'x_0+x_0^2)/2\hbar(t'-t_0)} e^{-x_0^2/2a^2}$$

$$= \sqrt{\frac{m}{2\pi\hbar i(t'-t_0)}} \frac{1}{\sqrt{\sqrt{\pi}a}} e^{imx'^2/2\hbar(t'-t_0)} \int_{-\infty}^{\infty} dx_0 \, e^{-Ax_0^2+Bx_0}$$

$$A = \frac{\hbar(t'-t_0) - ima^2}{2\hbar a^2(t'-t_0)}$$

$$B = -\frac{imx'}{\hbar(t'-t_0)}$$

$$\langle x|\psi(t)\rangle = \sqrt{\frac{m}{2\pi\hbar i(t'-t_0)}} \frac{1}{\sqrt{\sqrt{\pi}a}} e^{imx'^2/2\hbar(t'-t_0)} e^{B^2/4A} \sqrt{\frac{\pi}{A}}$$

$$= \sqrt{\frac{m}{2\pi\hbar i(t'-t_0)}} \frac{1}{\sqrt{\sqrt{\pi}a}} e^{imx'^2/2\hbar(t'-t_0)} e^{\frac{-m^2a^2x'^2}{2\hbar(t'-t_0)[\hbar(t'-t_0)-ima^2]}} \sqrt{\frac{2\pi\hbar a^2(t'-t_0)}{\hbar(t'-t_0)-ima^2}}$$

$$x' = x$$

$$t' - t_0 = t$$

$$\langle x|\psi(t)\rangle = \sqrt{\frac{m}{2\pi\hbar it}} \frac{1}{\sqrt{\sqrt{\pi}a}} e^{imx^2/2\hbar t} e^{\frac{-m^2a^2x^2}{2\hbar t[\hbar t-ima^2]}} \sqrt{\frac{2\pi\hbar a^2 t}{\hbar t - ima^2}}$$

$$= \sqrt{\frac{ma}{\sqrt{\pi}(i\hbar t + ma^2)}} e^{-x^2/2[(i\hbar t/m)+a^2)]}$$

$$\psi(x,t) = \frac{1}{\sqrt{\sqrt{\pi}[a + (i\hbar t/ma)]}} e^{-x^2/2a^2[1+(i\hbar t/ma^2)]}$$

Problem 8.2

Base Case:

$$\int_{-\infty}^{\infty} dy_1 \, e^{i[(y_2-y_1)^2+(y_1-y_0)^2]} = \sqrt{\frac{i\pi}{2}} e^{i(y_2-y_0)^2/2}$$

Assume:

$$\int_{-\infty}^{\infty} dy_1 \, ... \int_{-\infty}^{\infty} dy_{k-1} \, \exp\left[i \sum_{i=1}^{k} (y_i - y_{i-1})^2\right] = \sqrt{\frac{(i\pi)^{k-1}}{k}} e^{i(y_k-y_0)^2/k}$$

$$\int_{-\infty}^{\infty} dy_1 \, ... \int_{-\infty}^{\infty} dy_k \, \exp\left[i \sum_{i=1}^{k+1} (y_i - y_{i-1})^2\right] = \sqrt{\frac{(i\pi)^{k-1}}{k}} \int_{-\infty}^{\infty} dy_k \, e^{i[(y_{k+1}-y_k)^2+(y_k-y_0)^2/k]}$$

$$(y_{k+1} - y_k)^2 + (y_k - y_0)^2/k = \left(y_{k+1}^2 + \frac{y_0^2}{k}\right) - 2\left(y_{k+1} + \frac{y_0}{k}\right) y_k + \left(1 + \frac{1}{k}\right) y_k^2$$

$$\sqrt{\frac{(i\pi)^{k-1}}{k}} \int_{-\infty}^{\infty} dy_k \, e^{i[(y_{k+1}-y_k)^2+(y_k-y_0)^2/k]} = \sqrt{\frac{(i\pi)^{k-1}}{k}} e^{i\left(y_{k+1}^2+\frac{y_0^2}{k}\right)} \int_{-\infty}^{\infty} dy_k \, e^{-Ay_k^2+By_k}$$

$$A = -i\left(\frac{k+1}{k}\right)$$

$$B = -2i\left(y_{k+1} + \frac{y_0}{k}\right)$$

$$\frac{B^2}{4A} = -\frac{ik}{k+1}\left(y_{k+1} + \frac{y_0}{k}\right)^2$$

$$i\left(y_{k+1}^2 + \frac{y_0^2}{k}\right) - \frac{ik}{k+1}\left(y_{k+1} + \frac{y_0}{k}\right)^2 = \frac{i}{k+1}(y_{k+1} - y_0)^2$$

$$\sqrt{\frac{(i\pi)^{k-1}}{k}} e^{i\left(y_{k+1}^2+\frac{y_0^2}{k}\right)} \int_{-\infty}^{\infty} dy_k \, e^{-Ay_k^2+By_k} = \sqrt{\frac{i\pi k}{k+1}} \sqrt{\frac{(i\pi)^{k-1}}{k}} e^{\frac{i(y_{k+1}-y_0)^2}{k+1}}$$

$$\int_{-\infty}^{\infty} dy_1 \, ... \int_{-\infty}^{\infty} dy_k \, \exp\left[i \sum_{i=1}^{k+1} (y_i - y_{i-1})^2\right] = \sqrt{\frac{(i\pi)^k}{k+1}} e^{i(y_{k+1}-y_0)^2/(k+1)}$$

Induction:

$$\int_{-\infty}^{\infty} dy_1 \, ... \int_{-\infty}^{\infty} dy_{N-1} \, \exp\left[i \sum_{i=1}^{N} (y_i - y_{i-1})^2\right] = \sqrt{\frac{(i\pi)^{N-1}}{N}} e^{i(y_N-y_0)^2/N}$$

Problem 8.3

Harmonic oscillator:

$$L = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} m\omega^2 x^2$$

$$\langle x', t' | x_0, t_0 \rangle = \int_{x_0}^{x'} D[x(t)] e^{iS[x(t)]/\hbar}$$

Represent $x(t)$ as sum of classical path $x_{cl}(t)$ and deviation $y(t)$.

$$x(t) = x_{cl}(t) + y(t)$$

The amplitude can be written as:

$$\langle x', t', x_0, t_0 \rangle = e^{iS[x_{cl}(t)]/\hbar} \int_{t_0}^{t'} D[y(t)] e^{iS[y(t)]/\hbar} = F(t', t_0) e^{iS[x_{cl}(t)/\hbar]}$$

See Feynman and Hibbs, *Path Integrals and Quantum Mechanics*, Sections 3-5 and 3-6.

Classical path:

$$\ddot{x}_{cl} + \omega^2 x_{cl} = 0$$

$$x_{cl} = A \sin \omega t + B \cos \omega t$$

Boundary conditions:

$$x_{cl}(t_0) = x_0$$
$$x_{cl}(t') = x'$$

$$A \sin \omega t_0 + B \cos \omega t_0 = x_0$$
$$A \sin \omega t' + B \cos \omega t' = x'$$

$$x_{cl}(t) = \frac{x_0}{\sin \omega(t' - t_0)} \sin \omega(t' - t) + \frac{x'}{\sin \omega(t' - t_0)} \sin \omega(t - t_0)$$

$$\dot{x}_{cl}(t) = -\frac{\omega x_0}{\sin \omega(t' - t_0)} \cos \omega(t' - t) + \frac{\omega x'}{\sin \omega(t' - t_0)} \cos \omega(t - t_0)$$

$$L = \frac{m\omega^2}{2 \sin^2 \omega(t' - t_0)} \left[(x'^2 + x_0^2) \cos 2\omega(t - t_0) - 2x' x_0 \cos \omega(t' + t_0 - 2t) \right]$$

$$S = \int_{t_0}^{t'} L \, dt = \frac{m\omega}{2 \sin \omega(t' - t_0)} [(x'^2 + x_0^2) \cos \omega(t' - t_0) - 2x' x_0]$$

$$\langle x', t' | x_0, t_0 \rangle = f(t' - t_0) \exp \left\{ \frac{im\omega}{2\hbar \sin \omega(t' - t_0)} [(x'^2 + x_0^2) \cos \omega(t' - t_0) - 2x' x_0] \right\}$$

Problem 8.4 - SKIPPED

Problem 8.5

$$L = \frac{m\dot{x}^2}{2}$$

$$S = \int_0^{t'} dt \, \frac{mx'^2}{2t'^2} = \frac{mx'^2}{2t'} = \frac{mvx'}{2}$$

a)

$$\frac{S}{\hbar} = \frac{(9.109 \cdot 10^{-31})(2.188 \cdot 10^6)(0.5 \cdot 10^{-11})}{2(1.054 \cdot 10^{-34})} = 0.0473$$

b)

$$\frac{S}{\hbar} = \frac{(9.109 \cdot 10^{-31})(2.188 \cdot 10^{6})(0.01)}{2(1.054 \cdot 10^{-34})} = 9.45 \cdot 10^{7}$$

Classical mechanics gives an adequate description of electron motion across 1 cm. The large phase provides a very tight constraint which singles out the classical path.

Problem 8.6 - SKIPPED