Solutions - Chapter 7

Kevin S. Huang

Problem 7.1

$$\hat{a}\left|n\right\rangle = c_-\left|n-1\right\rangle$$
$$\left\langle n\right|\hat{a}^\dagger = \left\langle n-1\right| c_-^*$$

$$\langle n|\hat{a}^\dagger\hat{a}|n\rangle = \langle n|\hat{N}|n\rangle = n\left\langle n|n\right\rangle$$
$$\langle n|\hat{a}^\dagger\hat{a}|n\rangle = \langle n-1|c_-^*c_-|n-1\rangle = |c_-|^2\left\langle n-1|n-1\right\rangle$$

$$c_- = \sqrt{n}$$

Problem 7.2

$$\hat{a}^\dagger \rightarrow \begin{pmatrix} 0 & 0 & 0 & 0 & ... \\ \sqrt{1} & 0 & 0 & 0 & ... \\ 0 & \sqrt{2} & 0 & 0 & ... \\ 0 & 0 & \sqrt{3} & 0 & ... \\ ... & ... & ... & ... & ... \end{pmatrix}$$

$$\hat{a} \rightarrow \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & ... \\ 0 & 0 & \sqrt{2} & 0 & ... \\ 0 & 0 & 0 & \sqrt{3} & ... \\ 0 & 0 & 0 & 0 & ... \\ ... & ... & ... & ... & ... \end{pmatrix}$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger) \rightarrow \sqrt{\frac{\hbar}{2m\omega}} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & ... \\ \sqrt{1} & 0 & \sqrt{2} & 0 & ... \\ 0 & \sqrt{2} & 0 & \sqrt{3} & ... \\ 0 & 0 & \sqrt{3} & 0 & ... \\ ... & ... & ... & ... & ... \end{pmatrix}$$

$$\hat{p} = i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a}-\hat{a}^\dagger) \rightarrow i\sqrt{\frac{m\omega\hbar}{2}} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & ... \\ -\sqrt{1} & 0 & \sqrt{2} & 0 & ... \\ 0 & -\sqrt{2} & 0 & \sqrt{3} & ... \\ 0 & 0 & -\sqrt{3} & 0 & ... \\ ... & ... & ... & ... & ... \end{pmatrix}$$

$$\hat{x}\hat{p} = \frac{i\hbar}{2}\begin{pmatrix} \sqrt{0}^2-\sqrt{1}^2 & 0 & \sqrt{1}\sqrt{2} & 0 & ... \\ 0 & \sqrt{1}^2-\sqrt{2}^2 & 0 & \sqrt{2}\sqrt{3} & ... \\ -\sqrt{1}\sqrt{2} & 0 & \sqrt{2}^2-\sqrt{3}^2 & 0 & ... \\ 0 & -\sqrt{2}\sqrt{3} & 0 & \sqrt{3}^2-\sqrt{4}^2 & ... \\ ... & ... & ... & ... & ... \end{pmatrix}$$

$$\hat{p}\hat{x} = \frac{i\hbar}{2}\begin{pmatrix} \sqrt{1}^2-\sqrt{0}^2 & 0 & \sqrt{1}\sqrt{2} & 0 & ... \\ 0 & \sqrt{2}^2-\sqrt{1}^2 & 0 & \sqrt{2}\sqrt{3} & ... \\ -\sqrt{1}\sqrt{2} & 0 & \sqrt{3}^2-\sqrt{2}^2 & 0 & ... \\ 0 & -\sqrt{2}\sqrt{3} & 0 & \sqrt{4}^2-\sqrt{3}^2 & ... \\ ... & ... & ... & ... & ... \end{pmatrix}$$

$$\hat{x}\hat{p}-\hat{p}\hat{x} = \frac{i\hbar}{2}\begin{pmatrix} 2 & 0 & 0 & 0 & ... \\ 0 & 2 & 0 & 0 & ... \\ 0 & 0 & 2 & 0 & ... \\ 0 & 0 & 0 & 2 & ... \\ ... & ... & ... & ... & ... \end{pmatrix}$$

$$[\hat{x},\hat{p}] = i\hbar$$

Problem 7.3

Base Case:

$$|0\rangle = \frac{(\hat{a}^\dagger)^0}{\sqrt{0!}}|0\rangle = |0\rangle$$

Assume:

$$|k\rangle = \frac{(\hat{a}^\dagger)^k}{\sqrt{k!}}|0\rangle$$

$$\hat{a}^\dagger|k\rangle = \sqrt{k+1}\,|k+1\rangle$$

$$|k+1\rangle = \frac{\hat{a}^\dagger|k\rangle}{\sqrt{k+1}} = \frac{\hat{a}^\dagger(\hat{a}^\dagger)^k}{\sqrt{k+1}\sqrt{k!}}|0\rangle = \frac{(\hat{a}^\dagger)^{k+1}}{\sqrt{(k+1)!}}|0\rangle$$

By induction:

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle$$

Problem 7.4

$$\langle p|\hat{x}|\psi\rangle = i\hbar\frac{\partial}{\partial p}\langle p|\psi\rangle$$

$$\langle p|\hat{a}|0\rangle = 0$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right)$$

$$\langle p|\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}+\frac{i}{m\omega}\hat{p}\right)|0\rangle = 0$$

$$\langle p|\hat{x}|0\rangle + \frac{i}{m\omega}\langle p|\hat{p}|0\rangle = i\hbar\frac{\partial}{\partial p}\langle p|0\rangle + \frac{ip}{m\omega}\langle p|0\rangle = 0$$

$$\frac{d\psi}{dp} = -\frac{p}{m\hbar\omega}\psi$$

$$\int\frac{d\psi}{\psi} = \int -\frac{p}{m\hbar\omega}\,dp$$

$$\psi_0(p) = Ne^{-p^2/2m\hbar\omega}$$

$$\psi_0(p) = \left(\frac{1}{\pi m\hbar\omega}\right)^{1/4} e^{-p^2/2m\hbar\omega}$$

Problem 7.5

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^\dagger)$$
$$[\hat{a},\hat{a}^\dagger] = 1$$
$$\hat{a}\hat{a}^\dagger = \hat{N}+1$$

For energy eigenstates: $\langle x\rangle = 0$ and $\langle p\rangle = 0$

$$\Delta x^2 = \langle x^2\rangle$$
$$\Delta p^2 = \langle p^2\rangle$$

$$\Delta x^2 = \frac{\hbar}{2m\omega}\langle n|(\hat{a}+\hat{a}^\dagger)^2|n\rangle = \frac{\hbar}{2m\omega}\langle n|[\hat{a}^2+(\hat{a}^\dagger)^2+\hat{a}\hat{a}^\dagger+\hat{a}^\dagger\hat{a}]|n\rangle$$
$$= \frac{\hbar}{2m\omega}(\langle n|\hat{a}\hat{a}^\dagger|n\rangle + \langle n|\hat{a}^\dagger\hat{a}|n\rangle) = \frac{\hbar}{2m\omega}(\langle n|\hat{N}+1|n\rangle + \langle n|\hat{N}|n\rangle) = \frac{\hbar}{2m\omega}(2n+1)\langle n|n\rangle$$
$$\Delta x = \sqrt{\left(n+\frac{1}{2}\right)\frac{\hbar}{m\omega}}$$

$$\Delta p^2 = -\frac{m\omega\hbar}{2}\langle n|(\hat{a}-\hat{a}^\dagger)^2|n\rangle = -\frac{m\omega\hbar}{2}\langle n|[\hat{a}^2+(\hat{a}^\dagger)^2-\hat{a}\hat{a}^\dagger-\hat{a}^\dagger\hat{a}]|n\rangle$$
$$= \frac{m\omega\hbar}{2}(\langle n|\hat{a}\hat{a}^\dagger|n\rangle + \langle n|\hat{a}^\dagger\hat{a}|n\rangle) = \frac{m\omega\hbar}{2}(\langle n|\hat{N}+1|n\rangle + \langle n|\hat{N}|n\rangle) = \frac{m\omega\hbar}{2}(2n+1)\langle n|n\rangle$$
$$\Delta p = \sqrt{\left(n+\frac{1}{2}\right)m\omega\hbar}$$

Problem 7.6

$$\hat{p} = -i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a} - \hat{a}^\dagger)$$

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

$$|\alpha|^2 = \frac{1}{2}$$

$$|\beta|^2 = \frac{1}{2}$$

Choose $\alpha = \frac{1}{\sqrt{2}}$:

$$\beta = \frac{e^{i\theta}}{\sqrt{2}}$$

$$\langle p\rangle = \left(\frac{m\omega\hbar}{2}\right)^{1/2}$$

$$\langle p\rangle = \langle\psi|\hat{p}|\psi\rangle = (\alpha^*\,\langle 0| + \beta^*\,\langle 1|)\left[-i\sqrt{\frac{m\omega\hbar}{2}}(\hat{a} - \hat{a}^\dagger)\right](\alpha\,|0\rangle + \beta\,|1\rangle)$$

$$(\hat{a} - \hat{a}^\dagger)\,|0\rangle = -\,|1\rangle$$

$$(\hat{a} - \hat{a}^\dagger)\,|1\rangle = |0\rangle - \sqrt{2}\,|2\rangle$$

$$\langle p\rangle = -i\sqrt{\frac{m\omega\hbar}{2}}(\alpha^*\,\langle 0| + \beta^*\,\langle 1|)[-\alpha\,|1\rangle + \beta(|0\rangle - \sqrt{2}\,|2\rangle)] = -i\sqrt{\frac{m\omega\hbar}{2}}(-\alpha\beta^* + \alpha^*\beta)$$

$$-i\sqrt{\frac{m\omega\hbar}{2}}\left(-\frac{1}{\sqrt{2}}\frac{e^{-i\theta}}{\sqrt{2}} + \frac{1}{\sqrt{2}}\frac{e^{i\theta}}{\sqrt{2}}\right) = \sqrt{\frac{m\omega\hbar}{2}}$$

$$-i\left(\frac{e^{i\theta} - e^{-i\theta}}{2}\right) = 1$$

$$\sin\theta = 1$$

$$|\psi(0)\rangle = \frac{1}{\sqrt{2}}\,|0\rangle + \frac{i}{\sqrt{2}}\,|1\rangle$$

$$|\psi(t)\rangle = \frac{e^{-iE_0t/\hbar}}{\sqrt{2}}\,|0\rangle + \frac{ie^{-iE_1t//\hbar}}{\sqrt{2}}\,|1\rangle$$

$$\langle p\rangle(t) = -i\sqrt{\frac{m\omega\hbar}{2}}(-\alpha\beta^* + \alpha^*\beta) = -i\sqrt{\frac{m\omega\hbar}{2}}\left(\frac{e^{-iE_0t/\hbar}}{\sqrt{2}}\frac{ie^{iE_1t/\hbar}}{\sqrt{2}} + \frac{e^{iE_0t/\hbar}}{\sqrt{2}}\frac{ie^{-iE_1t/\hbar}}{\sqrt{2}}\right)$$

$$= \sqrt{\frac{m\omega\hbar}{2}} \left[\frac{e^{i(E_1-E_0)t/\hbar} + e^{-i(E_1-E_0)t/\hbar}}{2} \right] = \sqrt{\frac{m\omega\hbar}{2}} \cos \frac{(E_1 - E_0)t}{\hbar}$$

$$\langle p \rangle (t) = \sqrt{\frac{m\omega\hbar}{2}} \cos \omega t$$

Problem 7.7 - SKIPPED

Problem 7.8

$$|\psi(0)\rangle = c_n |n\rangle + c_{n+1} |n+1\rangle$$
$$|\psi(t)\rangle = e^{-i(n+1/2)\omega t}(c_n |n\rangle + c_{n+1}e^{-i\omega t} |n+1\rangle)$$
$$\langle \psi(t)| = e^{i(n+1/2)\omega t}(c_n^* \langle n| + c_{n+1}^* e^{i\omega t} \langle n+1|)$$

$$\langle x \rangle = \langle \psi | \hat{x} | \psi \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle \psi | (\hat{a} + \hat{a}^\dagger) | \psi \rangle$$

$$\hat{a}(c_n |n\rangle + c_{n+1}e^{-i\omega t} |n+1\rangle) = \sqrt{n} c_n |n-1\rangle + \sqrt{n+1} c_{n+1} e^{-i\omega t} |n\rangle$$
$$\hat{a}^\dagger(c_n |n\rangle + c_{n+1}e^{-i\omega t} |n+1\rangle) = \sqrt{n+1} c_n |n+1\rangle + \sqrt{n+2} c_{n+1} e^{-i\omega t} |n+2\rangle$$

$$\langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}} (c_n^* \langle n| + c_{n+1}^* e^{i\omega t} \langle n+1|)$$
$$(\sqrt{n} c_n |n-1\rangle + \sqrt{n+1} c_{n+1} e^{-i\omega t} |n\rangle + \sqrt{n+1} c_n |n+1\rangle + \sqrt{n+2} c_{n+1} e^{-i\omega t} |n+2\rangle)$$

$$\langle x \rangle = \sqrt{\frac{\hbar}{2m\omega}} (c_n^* \langle n| + c_{n+1}^* e^{i\omega t} \langle n+1|)(\sqrt{n+1} c_{n+1} e^{-i\omega t} |n\rangle + \sqrt{n+1} c_n |n+1\rangle)$$
$$= \sqrt{\frac{\hbar}{2m\omega}} \left(\sqrt{n+1} c_n^* c_{n+1} e^{-i\omega t} + \sqrt{n+1} c_n c_{n+1}^* e^{i\omega t} \right) = \sqrt{\frac{\hbar(n+1)}{2m\omega}} (c_n^* c_{n+1} e^{-i\omega t} + c_n c_{n+1}^* e^{i\omega t})$$

$$C_1 e^{i\omega t} + C_2 e^{-i\omega t} = (C_1 + C_2) \cos \omega t + (iC_1 - iC_2) \sin \omega t$$

$$\langle x \rangle = A \cos(\omega t + \delta) = (A \cos \delta) \cos \omega t + (-A \sin \delta) \sin \omega t$$

$$C_1 = \sqrt{\frac{\hbar(n+1)}{2m\omega}} c_n^* c_{n+1}$$
$$C_2 = \sqrt{\frac{\hbar(n+1)}{2m\omega}} c_n c_{n+1}^*$$
$$A = 2\sqrt{C_1 C_2}$$
$$\tan \delta = i \frac{C_2 - C_1}{C_2 + C_1}$$

$$\langle p \rangle = \langle \psi | \hat{p} | \psi \rangle = -i\sqrt{\frac{m\omega\hbar}{2}} \langle \psi | (\hat{a} - \hat{a}^\dagger) | \psi \rangle$$

$$\langle p \rangle = -i\sqrt{\frac{m\omega\hbar}{2}}(c_n^* \langle n| + c_{n+1}^* e^{i\omega t} \langle n+1|)$$

$$(\sqrt{n} c_n |n-1\rangle + \sqrt{n+1} c_{n+1} e^{-i\omega t} |n\rangle - \sqrt{n+1} c_n |n+1\rangle - \sqrt{n+2} c_{n+1} e^{-i\omega t} |n+2\rangle)$$

$$\langle p \rangle = -i\sqrt{\frac{m\omega\hbar}{2}}(c_n^* \langle n| + c_{n+1}^* e^{i\omega t} \langle n+1|)(\sqrt{n+1} c_{n+1} e^{-i\omega t} |n\rangle - \sqrt{n+1} c_n |n+1\rangle)$$

$$= -i\sqrt{\frac{m\omega\hbar}{2}}(\sqrt{n+1} c_n^* c_{n+1} e^{-i\omega t} - \sqrt{n+1} c_n c_{n+1}^* e^{i\omega t})$$

$$C_1 e^{i\omega t} + C_2 e^{-i\omega t} = (C_1 + C_2)\cos\omega t + (iC_1 - iC_2)\sin\omega t$$

$$\langle p \rangle = \langle \psi | \hat{p} | \psi \rangle = -m\omega A \sin(\omega t + \delta) = (-m\omega A \sin\delta)\cos\omega t + (-m\omega A \cos\delta)\sin\omega t$$

$$C_1' = i\sqrt{\frac{m\omega\hbar(n+1)}{2}} c_n^* c_{n+1} = im\omega C_1$$

$$C_2' = -i\sqrt{\frac{m\omega\hbar(n+1)}{2}} c_n c_{n+1}^* = -im\omega C_2$$

$$m^2\omega^2 A^2 = 4C_1' C_2' = 4m^2\omega^2 C_1^2 C_2^2$$

$$A = 2\sqrt{C_1 C_2}$$

$$\tan\delta = \frac{C_1' + C_2'}{i(C_1' - C_2')} = i\frac{C_2 - C_1}{C_2 + C_1}$$

Ehrenfest's theorem:

$$\frac{d\langle p \rangle}{dt} = -m\omega^2 A \cos(\omega t + \delta) = -m\omega^2 \langle x \rangle = \left\langle -\frac{dV}{dx} \right\rangle$$

$$\frac{d\langle x \rangle}{dt} = -\omega A \sin(\omega t + \delta) = \frac{\langle p \rangle}{m}$$

Problem 7.9 - SKIPPED

Problem 7.10

$$\hat{\Pi} |x\rangle = |-x\rangle$$

$$\langle x| \hat{\Pi}^\dagger = \langle -x|$$

$$\langle \psi | \hat{\Pi} | x \rangle = \langle \psi | -x \rangle = \psi^*(-x)$$

$$\langle x|\hat{\Pi}^\dagger|\psi\rangle = \langle -x|\psi\rangle = \psi(-x)$$

$$\langle\psi|\hat{\Pi}|x\rangle^* = \langle x|\hat{\Pi}|\psi\rangle = \langle x|\hat{\Pi}^\dagger|\psi\rangle$$

Hermitian:

$$\hat{\Pi}^\dagger = \hat{\Pi}$$

Problem 7.11

$$\psi(x) = Ne^{-ax^2}$$
$$\psi'(x) = -2aNxe^{-ax^2}$$
$$\psi''(x) = 2aNe^{-ax^2}(2ax^2 - 1)$$

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\langle x|E\rangle + \frac{1}{2}m\omega^2x^2\langle x|E\rangle = E\langle x|E\rangle$$
$$-\frac{\hbar^2}{2m}2aNe^{-ax^2}(2ax^2 - 1) + \frac{1}{2}m\omega^2x^2Ne^{-ax^2} = ENe^{-ax^2}$$
$$-\frac{a\hbar^2}{m}(2ax^2 - 1) + \frac{1}{2}m\omega^2x^2 = E$$
$$\left(\frac{1}{2}m\omega^2 - \frac{2a^2\hbar^2}{m}\right)x^2 + \left(\frac{a\hbar^2}{m} - E\right) = 0$$
$$a = \frac{m\omega}{2\hbar}$$
$$E_e = \frac{a\hbar^2}{m} = \frac{\hbar\omega}{2}$$

Problem 7.12

Ground state:

$$|\psi\rangle = |0\rangle$$

$$\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$$

$$V = \frac{1}{2}m\omega^2x^2 = \frac{\hbar\omega}{2} = E_0$$

Turning point:

$$x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

Probability of particle located in classically disallowed region $V(x) > E$:

$$P = 2\int_{x_0}^{\infty} dx\,|\psi(x)|^2 = 2\sqrt{\frac{m\omega}{\pi\hbar}}\int_{x_0}^{\infty} dx\,e^{-m\omega x^2/\hbar}$$

$$u = x\sqrt{\frac{m\omega}{\hbar}}$$

$$1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt$$

$$P = \frac{2}{\sqrt{\pi}} \int_1^\infty du\, e^{-u^2} = 1 - \mathrm{erf}(1) \approx 0.157$$