Solutions - Chapter 4

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Problem 4.1

$$\hat{U}(dt) = 1 - \frac{i}{\hbar}\hat{H}dt$$

Unitary:

$$\hat{U}^{\dagger}(t)\hat{U}(t) = 1$$

Neglect $(dt)^2$ term:

$$\left(1 + \frac{i}{\hbar}\hat{H}^{\dagger}dt\right)\left(1 - \frac{i}{\hbar}\hat{H}dt\right) = 1 + \frac{i}{\hbar}\hat{H}^{\dagger}dt - \frac{i}{\hbar}\hat{H}dt = 1$$

$$\hat{H} = \hat{H}^{\dagger}$$

Problem 4.2

$$\hat{U}(t) = \lim_{n\to\infty}\left[1 - \frac{i}{\hbar}\hat{H}_1dt\right]\left[1 - \frac{i}{\hbar}\hat{H}_2dt\right]...\left[1 - \frac{i}{\hbar}\hat{H}_ndt\right]$$

$$\left[1 - \frac{i}{\hbar}\hat{H}_1dt\right]\left[1 - \frac{i}{\hbar}\hat{H}_2dt\right] = \left[1 - \frac{i}{\hbar}(\hat{H}_1 + \hat{H}_2)dt + \left(\frac{i}{\hbar}\right)^2\hat{H}_1dt\hat{H}_2dt\right]$$

$$\left[1 - \frac{i}{\hbar}\hat{H}_1dt\right]\left[1 - \frac{i}{\hbar}\hat{H}_2dt\right]\left[1 - \frac{i}{\hbar}\hat{H}_3dt\right] =$$

$$\left[1 - \frac{i}{\hbar}(\hat{H}_1 + \hat{H}_2 + \hat{H}_3)dt + \left(\frac{i}{\hbar}\right)^2\left[\hat{H}_1dt(\hat{H}_2 + \hat{H}_3)dt + \hat{H}_2dt\hat{H}_3dt\right] - \left(\frac{i}{\hbar}\right)^3\hat{H}_3dt\hat{H}_2dt\hat{H}_1dt\right]$$

$$\hat{U}(t) = 1 + \left(-\frac{i}{\hbar}\right)\int_0^t \hat{H}(t_1)\,dt_1 + \left(-\frac{i}{\hbar}\right)^2\int_0^t \hat{H}(t_1)\,dt_1\int_0^{t_1}\hat{H}(t_2)\,dt_2 + ...$$

$$\hat{U_n} = \left(-\frac{i}{\hbar}\right)^n\int_0^t \hat{H}(t_1)\,dt_1\int_0^{t_1}\hat{H}(t_2)\,dt_2...\int_0^{t_{n-1}}\hat{H}(t_n)\,dt_n$$

$[\hat{H}(t_1), \hat{H}(t_2)] = 0$:

$$\hat{U_n} = \frac{1}{n!}\left(-\frac{i}{\hbar}\right)^n\left(\int_0^t \hat{H}(t')\,dt'\right)^n$$

$$\hat{U}(t) = \exp\left[-\frac{i}{\hbar}\int_0^t dt'\hat{H}(t')\right]$$

Problem 4.3
Time-independent observable:

$$\frac{d}{dt}\langle A\rangle = \frac{i}{\hbar}\langle\psi(t)|[\hat{H},\hat{A}]|\psi(t)\rangle$$

$$\hat{H}|E\rangle = E|E\rangle$$

$$\langle E|[\hat{H},\hat{A}]|E\rangle = \langle E|\hat{H}\hat{A}-\hat{A}\hat{H}|E\rangle = \langle E|\hat{H}\hat{A}|E\rangle - \langle E|\hat{A}\hat{H}|E\rangle$$

$$\frac{d}{dt}\langle A\rangle = \langle E|E\hat{A}|E\rangle - \langle E|\hat{A}E|E\rangle = E\langle E|\hat{A}|E\rangle - E\langle E|\hat{A}|E\rangle = 0$$

Problem 4.4

$$\hat{\mu} = -\frac{gq}{2mc}\hat{S}$$

$$\vec{B} = B_0\hat{\mathbf{i}}$$

$$\hat{H} = -\hat{\mu}\cdot\vec{B} = \frac{ge}{2mc}\hat{S}_xB_0 = \omega_0\hat{S}_x$$

$$\hat{U}(t) = e^{-i\hat{H}t/\hbar} = e^{-i\omega_0\hat{S}_xt/\hbar} = e^{-i\hat{S}_x\phi/\hbar} = \hat{R}(\phi\hat{\mathbf{i}})$$

$$|\langle -\mathbf{z}|\hat{R}(\phi\hat{\mathbf{i}})|+\mathbf{z}\rangle|^2 = \frac{1}{4}$$

$$|\langle -\mathbf{z}|\left[\cos\left(\frac{\phi}{2}\right)|+\mathbf{z}\rangle - i\sin\left(\frac{\phi}{2}\right)|-\mathbf{z}\rangle\right]\rangle| = \frac{1}{2}$$

$$\sin\left(\frac{\phi}{2}\right) = \frac{1}{2}$$

$$\phi = \frac{\pi}{3}$$

$$t = \frac{l_0}{v_0}$$

$$\omega_0\frac{l_0}{v_0} = \frac{\pi}{3}$$

$$l_0 = \frac{\pi v_0}{3\omega_0}$$

Problem 4.5

$$\vec{B} = B_0 \sin\theta \hat{\mathbf{i}} + B_0 \cos\theta \hat{\mathbf{z}}$$

$$\hat{H} = \frac{\omega_0}{B_0} \hat{S} \cdot \vec{B} = \omega_0 (\hat{S}_x \sin\theta + \hat{S}_z \cos\theta) = \omega_0 \hat{S}_n \quad (\phi = 0)$$

Below we use ϕ as defined by $\omega_0 t$:

$$\hat{U}(t) = e^{-i\phi \hat{S_n}/\hbar} = 1 + \left(-\frac{i\phi}{\hbar}\right) \hat{S_n} + \frac{1}{2!}\left(-\frac{i\phi}{\hbar}\right)^2 \hat{S_n}^2 + ...$$

$$= 1 - \frac{1}{2!}\left(\frac{\phi}{2}\right)^2 + ... + (-i\sigma_n)\left[\left(\frac{\phi}{2}\right) - \frac{1}{3!}\left(\frac{\phi}{2}\right)^3 + ...\right] = \cos\left(\frac{\phi}{2}\right) - i\sigma_n \sin\left(\frac{\phi}{2}\right)$$

$$\sigma_n \underset{S_z}{\rightarrow} \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

$$\hat{U}(t) \underset{S_z}{\rightarrow} \begin{pmatrix} \cos(\phi/2) - i\sin(\phi/2)\cos\theta & -i\sin(\phi/2)\sin\theta \\ -i\sin(\phi/2)\sin\theta & \cos(\phi/2) + i\sin(\phi/2)\cos\theta \end{pmatrix}$$

$$\langle +\mathbf{y}| = \frac{1}{\sqrt{2}} \langle +\mathbf{z}| - \frac{i}{\sqrt{2}} \langle -\mathbf{z}|$$

$$\hat{U}(t) |+\mathbf{z}\rangle = \left[\cos\left(\frac{\phi}{2}\right) - i\sin\left(\frac{\phi}{2}\right)\cos\theta\right] |+\mathbf{z}\rangle - i\sin\left(\frac{\phi}{2}\right)\sin\theta\, |-\mathbf{z}\rangle$$

$$\langle +\mathbf{y}|\hat{U}(t)| + \mathbf{z}\rangle = \frac{1}{\sqrt{2}}\left[\cos\left(\frac{\phi}{2}\right) - i\sin\left(\frac{\phi}{2}\right)\cos\theta\right] - \frac{1}{\sqrt{2}}\sin\left(\frac{\phi}{2}\right)\sin\theta$$

$$P\left(S_y = \frac{\hbar}{2}\right) = |\langle +\mathbf{y}|\hat{U}(T)| + \mathbf{z}\rangle|^2 = \frac{1 - \sin(\omega_0 T)\sin\theta}{2}$$

$\theta = 0$:

$$P = \frac{1}{2}$$

$\theta = \pi/2$:

$$P = \frac{1 - \sin(\omega_0 T = \frac{\pi}{2})}{2} = 0$$

Problem 4.6

$$|\psi(t)\rangle = \frac{e^{-i\omega_0 t/2}}{\sqrt{2}} |+\mathbf{z}\rangle + \frac{e^{i\omega_0 t/2}}{\sqrt{2}} |-\mathbf{z}\rangle$$

$$\langle S_z \rangle = 0$$

$$\frac{d}{dt}\langle S_z\rangle = \frac{i}{\hbar}\langle\psi(t)|[\hat{H}, \hat{S}_z]|\psi(t)\rangle + \langle\psi(t)|\frac{\partial \hat{S}_z}{\partial t}|\psi(t)\rangle$$

$$\frac{\partial \hat{S}_z}{\partial t} = 0$$

$\hat{H} = \omega_0 \hat{S}_z$:

$$[\hat{H}, \hat{S}_z] = 0$$

$$\langle S_x \rangle = \frac{\hbar}{2} \cos \omega_0 t$$

$$\frac{d}{dt}\langle S_x \rangle = \frac{i}{\hbar} \langle \psi(t)|[\hat{H}, \hat{S}_x]|\psi(t)\rangle + \langle \psi(t)|\frac{\partial \hat{S}_x}{\partial t}|\psi(t)\rangle$$

$$\frac{\partial \hat{S}_x}{\partial t} = 0$$

$\hat{H} = \omega_0 \hat{S}_z$:

$$[\hat{H}, \hat{S}_x] = \frac{\omega_0 \hbar^2}{4}[\sigma_z, \sigma_x] = \frac{\omega_0 \hbar^2}{4}(2i\sigma_y) = \omega_0 \hbar i \hat{S}_y$$

$$\frac{d}{dt}\langle S_x \rangle = -\omega_0 \langle \psi(t)|\hat{S}_y|\psi(t)\rangle$$

$$\hat{S}_y |\psi(t)\rangle = \frac{i\hbar}{2} \left(\frac{e^{-i\omega_0 t/2}}{\sqrt{2}} |-\mathbf{z}\rangle - \frac{e^{i\omega_0 t/2}}{\sqrt{2}} |+\mathbf{z}\rangle \right)$$

$$-\omega_0 \langle \psi(t)|\hat{S}_y|\psi(t)\rangle = \frac{-\omega_0 i\hbar}{2} \left(\frac{e^{i\omega_0 t/2}}{\sqrt{2}} \langle +\mathbf{z}| + \frac{e^{-i\omega_0 t/2}}{\sqrt{2}} \langle -\mathbf{z}| \right) \left(-\frac{e^{i\omega_0 t/2}}{\sqrt{2}} |+\mathbf{z}\rangle + \frac{e^{-i\omega_0 t/2}}{\sqrt{2}} |-\mathbf{z}\rangle \right)$$

$$= \frac{-\omega_0 i\hbar}{2} \left(\frac{-e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} \right) = \frac{-\omega_0 i\hbar}{2} \left(\frac{-\cos \omega_0 t - i \sin \omega_0 t + \cos \omega_0 t - i \sin \omega_0 t}{2} \right) = -\frac{\omega_0 \hbar}{2} \sin \omega_0 t$$

$$\frac{d}{dt}\langle \hat{S}_x \rangle = \frac{d}{dt} \left(\frac{\hbar}{2} \cos \omega_0 t \right) = -\frac{\omega_0 \hbar}{2} \sin \omega_0 t$$

$$\langle S_y \rangle = \frac{\hbar}{2} \sin \omega_0 t$$

$$\frac{d}{dt}\langle S_y \rangle = \frac{i}{\hbar} \langle \psi(t)|[\hat{H}, \hat{S}_y]|\psi(t)\rangle + \langle \psi(t)|\frac{\partial \hat{S}_y}{\partial t}|\psi(t)\rangle$$

$$\frac{\partial \hat{S}_y}{\partial t} = 0$$

$\hat{H} = \omega_0 \hat{S}_z$:

$$[\hat{H}, \hat{S}_x] = \frac{\omega_0 \hbar^2}{4}[\sigma_z, \sigma_y] = \frac{\omega_0 \hbar^2}{4}(-2i\sigma_x) = -\omega_0 \hbar i \hat{S}_x$$

$$\frac{d}{dt}\langle S_y \rangle = \omega_0 \langle \psi(t)|\hat{S}_x|\psi(t)\rangle$$

$$\hat{S}_x |\psi(t)\rangle = \frac{\hbar}{2}\left(\frac{e^{-i\omega_0 t/2}}{\sqrt{2}}|-\mathbf{z}\rangle + \frac{e^{i\omega_0 t/2}}{\sqrt{2}}|+\mathbf{z}\rangle\right)$$

$$\omega_0 \langle \psi(t)|\hat{S}_x|\psi(t)\rangle = \frac{\omega_0 \hbar}{2}\left(\frac{e^{i\omega_0 t/2}}{\sqrt{2}}\langle +\mathbf{z}| + \frac{e^{-i\omega_0 t/2}}{\sqrt{2}}\langle -\mathbf{z}|\right)\left(\frac{e^{i\omega_0 t/2}}{\sqrt{2}}|+\mathbf{z}\rangle + \frac{e^{-i\omega_0 t/2}}{\sqrt{2}}|-\mathbf{z}\rangle\right)$$

$$= \frac{\omega_0 \hbar}{2}\left(\frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2}\right) = \frac{\omega_0 \hbar}{2}\left(\frac{\cos\omega_0 t + i\sin\omega_0 t + \cos\omega_0 t - i\sin\omega_0 t}{2}\right) = \frac{\omega_0 \hbar}{2}\cos\omega_0 t$$

$$\frac{d}{dt}\langle \hat{S}_y \rangle = \frac{d}{dt}\left(\frac{\hbar}{2}\sin\omega_0 t\right) = \frac{\omega_0 \hbar}{2}\cos\omega_0 t$$

Problem 4.7 - Skipped

Problem 4.8

$$|\psi(0)\rangle = \cos\frac{\theta}{2}|+\mathbf{z}\rangle + \sin\frac{\theta}{2}|-\mathbf{z}\rangle$$

$$|\psi(t)\rangle = e^{-i\omega_0 t/2}\cos\frac{\theta}{2}|+\mathbf{z}\rangle + e^{i\omega_0 t/2}\sin\frac{\theta}{2}|-\mathbf{z}\rangle$$

$$\langle S_x \rangle = \langle \psi(t)|\hat{S}_x|\psi(t)\rangle$$

$$= \left(e^{i\omega_0 t/2}\cos\frac{\theta}{2}\langle +\mathbf{z}| + e^{-i\omega_0 t/2}\sin\frac{\theta}{2}\langle -\mathbf{z}|\right)\hat{S}_x\left(e^{-i\omega_0 t/2}\cos\frac{\theta}{2}|+\mathbf{z}\rangle + e^{i\omega_0 t/2}\sin\frac{\theta}{2}|-\mathbf{z}\rangle\right)$$

$$= \frac{\hbar}{2}\left(e^{i\omega_0 t/2}\cos\frac{\theta}{2}\langle +\mathbf{z}| + e^{-i\omega_0 t/2}\sin\frac{\theta}{2}\langle -\mathbf{z}|\right)\left(e^{-i\omega_0 t/2}\cos\frac{\theta}{2}|-\mathbf{z}\rangle + e^{i\omega_0 t/2}\sin\frac{\theta}{2}|+\mathbf{z}\rangle\right)$$

$$= \frac{\hbar}{2}\left(e^{i\omega_0 t}\sin\frac{\theta}{2}\cos\frac{\theta}{2} + e^{-i\omega_0 t}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) = \frac{\hbar}{2}\sin\frac{\theta}{2}\cos\frac{\theta}{2}(2\cos\omega_0 t)$$

$$\langle S_x \rangle = \frac{\hbar}{2}\sin\theta\cos\omega_0 t$$

$$\langle S_y \rangle = \langle \psi(t)|\hat{S}_y|\psi(t)\rangle$$

$$= \left(e^{i\omega_0 t/2} \cos\frac{\theta}{2} \langle +\mathbf{z}| + e^{-i\omega_0 t/2} \sin\frac{\theta}{2} \langle -\mathbf{z}| \right) \hat{S}_y \left(e^{-i\omega_0 t/2} \cos\frac{\theta}{2} |+\mathbf{z}\rangle + e^{-i\omega_0 t/2} \sin\frac{\theta}{2} |-\mathbf{z}\rangle \right)$$

$$= \frac{i\hbar}{2} \left(e^{i\omega_0 t/2} \cos\frac{\theta}{2} \langle +\mathbf{z}| + e^{-i\omega_0 t/2} \sin\frac{\theta}{2} \langle -\mathbf{z}| \right) \left(e^{-i\omega_0 t/2} \cos\frac{\theta}{2} |-\mathbf{z}\rangle - e^{i\omega_0 t/2} \sin\frac{\theta}{2} |+\mathbf{z}\rangle \right)$$

$$= \frac{i\hbar}{2} \left(e^{-i\omega_0 t} \sin\frac{\theta}{2} \cos\frac{\theta}{2} - e^{i\omega_0 t} \sin\frac{\theta}{2} \cos\frac{\theta}{2} \right) = \frac{i\hbar}{2} \sin\frac{\theta}{2} \cos\frac{\theta}{2} (-2i \sin\omega_0 t)$$

$$\langle S_y \rangle = \frac{\hbar}{2} \sin\theta \sin\omega_0 t$$

$$\langle S_z \rangle = \langle \psi(t) | \hat{S}_z | \psi(t) \rangle$$

$$= \left(e^{i\omega_0 t/2} \cos\frac{\theta}{2} \langle +\mathbf{z}| + e^{-i\omega_0 t/2} \sin\frac{\theta}{2} \langle -\mathbf{z}| \right) \hat{S}_z \left(e^{-i\omega_0 t/2} \cos\frac{\theta}{2} |+\mathbf{z}\rangle + e^{-i\omega_0 t/2} \sin\frac{\theta}{2} |-\mathbf{z}\rangle \right)$$

$$= \frac{\hbar}{2} \left(e^{i\omega_0 t/2} \cos\frac{\theta}{2} \langle +\mathbf{z}| + e^{-i\omega_0 t/2} \sin\frac{\theta}{2} \langle -\mathbf{z}| \right) \left(e^{-i\omega_0 t/2} \cos\frac{\theta}{2} |+\mathbf{z}\rangle - e^{i\omega_0 t/2} \sin\frac{\theta}{2} |-\mathbf{z}\rangle \right)$$

$$= \frac{\hbar}{2} \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \right) = \frac{\hbar}{2} \cos\theta$$

$$\langle S_z \rangle = \frac{\hbar}{2} \cos\theta$$

Problem 4.9

$$\hat{H} = \omega_0 \hat{S}_z + \omega_1 (\cos\omega t) \hat{S}_x$$

$$|\psi(0)\rangle = |+\mathbf{z}\rangle$$

$$\hat{H} |\psi(t)\rangle = i\hbar \frac{d\,|\psi(t)\rangle}{dt}$$

$$\frac{\hbar}{2} \begin{pmatrix} \omega_0 & \omega_1 \cos\omega t \\ \omega_1 \cos\omega t & -\omega_0 \end{pmatrix} \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = i\hbar \begin{pmatrix} \dot{a}(t) \\ \dot{b}(t) \end{pmatrix}$$

Approximation: $B_1 \ll B_0, \omega_1 \ll \omega_0$

$$\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} c(t) e^{-i\omega_0 t/2} \\ d(t) e^{i\omega_0 t/2} \end{pmatrix}$$

Approximation: $\omega \approx \omega_0$

$$i\dot{c}(t) = \frac{\omega_1}{4} e^{i(\omega_0 - \omega)t} d(t)$$
$$i\dot{d}(t) = \frac{\omega_1}{4} e^{i(\omega - \omega_0)t} c(t)$$

$$i\ddot{c}(t) = \frac{\omega_1}{4}[i(\omega_0 - \omega)e^{i(\omega_0-\omega)t} d(t) + e^{i(\omega_0-\omega)t}\dot{d}(t)]$$
$$i\ddot{d}(t) = \frac{\omega_1}{4}[i(\omega - \omega_0)e^{i(\omega-\omega_0)t} c(t) + e^{i(\omega-\omega_0)t}\dot{c}(t)]$$

$$i\ddot{c}(t) = \frac{\omega_1}{4}\left[i(\omega_0 - \omega)e^{i(\omega_0-\omega)t}\frac{4i}{\omega_1}e^{i(\omega-\omega_0)t}\dot{c}(t) - e^{i(\omega_0-\omega)t}\frac{i\omega_1}{4}e^{i(\omega-\omega_0)t}c(t)\right]$$

$$\ddot{c}(t) = \frac{\omega_1}{4}\left[(\omega_0 - \omega)\frac{4i}{\omega_1}\dot{c}(t) - \frac{\omega_1}{4}c(t)\right]$$

$$\ddot{c}(t) = i(\omega_0 - \omega)\dot{c}(t) - \left(\frac{\omega_1}{4}\right)^2 c(t)$$

$$\ddot{c} - i(\omega_0 - \omega)\dot{c} + \left(\frac{\omega_1}{4}\right)^2 c = 0$$

Characteristic equation:

$$r^2 + [i(\omega - \omega_0)]r + \left(\frac{\omega_1}{4}\right)^2 = 0$$

$$r = \frac{i(\omega_0 - \omega) \pm i\sqrt{(\omega_0 - \omega)^2 + ({\omega_1}^2/4)}}{2}$$

$$c(t) = e^{i(\omega_0-\omega)t/2}\left(A\sin\frac{\sqrt{(\omega_0 - \omega)^2 + ({\omega_1}^2/4)}}{2}t + B\cos\frac{\sqrt{(\omega_0 - \omega)^2 + ({\omega_1}^2/4)}}{2}t\right)$$

$c(0) = 1, d(0) = 0$:

$$A = \frac{i(\omega - \omega_0)}{\sqrt{(\omega_0 - \omega)^2 + ({\omega_1}^2/4)}}$$
$$B = 1$$

$$c(t) = e^{i(\omega_0-\omega)t/2}\left(\frac{i(\omega - \omega_0)}{\sqrt{(\omega_0 - \omega)^2 + ({\omega_1}^2/4)}}\sin\frac{\sqrt{(\omega_0 - \omega)^2 + ({\omega_1}^2/4)}}{2}t + \cos\frac{\sqrt{(\omega_0 - \omega)^2 + ({\omega_1}^2/4)}}{2}t\right)$$

$$\dot{c}(t) = e^{i(\omega_0-\omega)t/2}\left(\frac{{\omega_1}^2/4}{2\sqrt{(\omega_0 - \omega)^2 + ({\omega_1}^2/4)}}\sin\frac{\sqrt{(\omega_0 - \omega)^2 + ({\omega_1}^2/4)}}{2}t\right)$$

$$d(t) = ie^{-i(\omega_0-\omega)t/2}\left(\frac{\omega_1/2}{\sqrt{(\omega_0-\omega)^2+({\omega_1}^2/4)}}\sin\frac{\sqrt{(\omega_0-\omega)^2+({\omega_1}^2/4)}}{2}t\right)$$

$$|\langle -\mathbf{z}|\psi(t)\rangle|^2 = b^*(t)b(t) = d^*(t)d(t) = \frac{{\omega_1}^2/4}{(\omega_0-\omega)^2+({\omega_1}^2/4)}\sin^2\frac{\sqrt{(\omega_0-\omega)^2+({\omega_1}^2/4)}}{2}t$$

Problem 4.10

$$|I\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|2\rangle$$

$$|II\rangle = \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|2\rangle$$

$$\langle 1|I\rangle = \frac{1}{\sqrt{2}}, \langle 2|I\rangle = \frac{1}{\sqrt{2}}$$

$$\langle 1|II\rangle = \frac{1}{\sqrt{2}}, \langle 2|II\rangle = -\frac{1}{\sqrt{2}}$$

$$\langle I|\hat{H}|I\rangle = \sum_{1,2}\langle I|a\rangle\langle a|\hat{H}|a'\rangle\langle a'|I\rangle = \frac{\langle 1|\hat{H}|1\rangle + \langle 1|\hat{H}|2\rangle + \langle 2|\hat{H}|1\rangle + \langle 2|\hat{H}|2\rangle}{2} = E_0 - A$$

$$\langle I|\hat{H}|II\rangle = \frac{\langle 1|\hat{H}|1\rangle - \langle 1|\hat{H}|2\rangle + \langle 2|\hat{H}|1\rangle - \langle 2|\hat{H}|2\rangle}{2} = \mu_e|\vec{E}|$$

$$\langle II|\hat{H}|I\rangle = \frac{\langle 1|\hat{H}|1\rangle + \langle 1|\hat{H}|2\rangle - \langle 2|\hat{H}|1\rangle - \langle 2|\hat{H}|2\rangle}{2} = \mu_e|\vec{E}|$$

$$\langle II|\hat{H}|II\rangle = \frac{\langle 1|\hat{H}|1\rangle - \langle 1|\hat{H}|2\rangle - \langle 2|\hat{H}|1\rangle + \langle 2|\hat{H}|2\rangle}{2} = E_0 + A$$

$$\hat{H} \underset{I,II}{\longrightarrow} \begin{pmatrix} E_0 - A & \mu_e|\vec{E}_0|\cos\omega t \\ \mu_e|\vec{E}_0|\cos\omega t & E_0 + A \end{pmatrix}$$

Compare with:

$$\hat{H} \underset{+z,-z}{\longrightarrow} \frac{\hbar}{2}\begin{pmatrix} \omega_0 & \omega_1\cos\omega t \\ \omega_1\cos\omega t & -\omega_0 \end{pmatrix}$$

Analogous Hamiltonian:

$$E_+ = E_0 + A$$

$$E_- = E_0 - A$$

$$\hat{H} \underset{II,I}{\rightarrow} \begin{pmatrix} E_+ & \mu_e|\vec{E}_0|\cos\omega t \\ \mu_e|\vec{E}_0|\cos\omega t & E_- \end{pmatrix}$$

$$i\begin{pmatrix} \dot{c}(t) \\ \dot{d}(t) \end{pmatrix} = \frac{\mu_e|\vec{E}_0|}{\hbar}\cos\omega t \begin{pmatrix} d(t)e^{i(E_+-E_-)t/\hbar} \\ c(t)e^{-i(E_+-E_-)t/\hbar} \end{pmatrix}$$

$$+z \rightarrow II, -z \rightarrow I$$

$$E_+ = \frac{\hbar\omega_0}{2} \rightarrow E_0 + A$$

$$E_- = -\frac{\hbar\omega_0}{2} \rightarrow E_0 - A$$

$$E_+ - E_- = \hbar\omega_0 \rightarrow 2A$$

$$\frac{\hbar\omega_1}{2} \rightarrow \mu_e|\vec{E}_0|$$

Analogue of Rabi's formula:

$$\psi(0) = |II\rangle$$

$$|\langle I|\psi(t)\rangle|^2 = \frac{{\mu_e}^2|\vec{E}_0|^2/\hbar^2}{(2A/\hbar-\omega)^2 + {\mu_e}^2|\vec{E}_0|^2/\hbar^2}\sin^2\frac{\sqrt{(2A/\hbar-\omega)^2 + {\mu_e}^2|\vec{E}_0|^2/\hbar^2}}{2}t$$

Problem 4.11

$$\vec{\mu} = \left(\frac{gq}{2mc}\right)\vec{S}$$

$$\vec{B} = B_0\hat{\mathbf{k}}$$

$$\hat{H} = -\vec{\mu}\cdot\vec{B} = -\frac{gq}{2mc}B_0\hat{S}_z = \omega_0\hat{S}_z$$

$$\hat{H}|1,1\rangle = \omega_0\hat{S}_z|1,1\rangle = \hbar\omega_0|1,1\rangle = E_1|1,1\rangle$$
$$\hat{H}|1,0\rangle = \omega_0\hat{S}_z|1,0\rangle = 0\omega_0|1,0\rangle = E_0|1,0\rangle$$
$$\hat{H}|1,-1\rangle = \omega_0\hat{S}_z|1,-1\rangle = -\hbar\omega_0|1,-1\rangle = E_{-1}|1,-1\rangle$$

$$|\psi(0)\rangle = |1,1\rangle_y = \frac{1}{2}|1,1\rangle + \frac{i\sqrt{2}}{2}|1,0\rangle - \frac{1}{2}|1,-1\rangle$$

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}\left(\frac{1}{2}|1,1\rangle + \frac{i\sqrt{2}}{2}|1,0\rangle - \frac{1}{2}|1,-1\rangle\right)$$

$$= \frac{e^{-iE_1 t/\hbar}}{2} |1,1\rangle + \frac{i\sqrt{2}e^{-iE_0 t/\hbar}}{2} |1,0\rangle - \frac{e^{-iE_{-1} t/\hbar}}{2} |1,-1\rangle$$

$$|\psi(t)\rangle = \frac{e^{-i\omega_0 t}}{2} |1,1\rangle + \frac{i\sqrt{2}}{2} |1,0\rangle - \frac{e^{i\omega_0 t}}{2} |1,-1\rangle$$

$$|1,1\rangle_x = \frac{1}{2} |1,1\rangle + \frac{\sqrt{2}}{2} |1,0\rangle + \frac{1}{2} |1,-1\rangle$$

$$|1,0\rangle_x = \frac{\sqrt{2}}{2} |1,1\rangle - \frac{\sqrt{2}}{2} |1,-1\rangle$$

$$|1,-1\rangle_x = \frac{1}{2} |1,1\rangle - \frac{\sqrt{2}}{2} |1,0\rangle + \frac{1}{2} |1,-1\rangle$$

$$|1,1\rangle_y = \frac{1}{2} |1,1\rangle + \frac{i\sqrt{2}}{2} |1,0\rangle - \frac{1}{2} |1,-1\rangle$$

$$|1,0\rangle_y = \frac{\sqrt{2}}{2} |1,1\rangle + \frac{\sqrt{2}}{2} |1,-1\rangle$$

$$|1,-1\rangle_y = \frac{1}{2} |1,1\rangle - \frac{i\sqrt{2}}{2} |1,0\rangle - \frac{1}{2} |1,-1\rangle$$

$$|\langle 1,1|_x|\psi(t)\rangle|^2 = \left|\frac{e^{-i\omega_0 t} - e^{i\omega_0 t}}{4} + \frac{i}{2}\right|^2 = \frac{(1-\sin\omega_0 t)^2}{4}$$

$$|\langle 1,0|_x|\psi(t)\rangle|^2 = \left|\frac{(e^{-i\omega_0 t} + e^{i\omega_0 t})\sqrt{2}}{4}\right|^2 = \frac{\cos^2\omega_0 t}{2}$$

$$|\langle 1,-1|_x|\psi(t)\rangle|^2 = \left|\frac{e^{-i\omega_0 t} - e^{i\omega_0 t}}{4} - \frac{i}{2}\right|^2 = \frac{(1+\sin\omega_0 t)^2}{4}$$

$$\langle S_x\rangle = \frac{(1-\sin\omega_0 t)^2}{4}(\hbar) + 0 + \frac{(1+\sin\omega_0 t)^2}{4}(-\hbar) = -\hbar\sin\omega_0 t$$

$$|\langle 1,1|_y|\psi(t)\rangle|^2 = \left|\frac{e^{-i\omega_0 t} + e^{i\omega_0 t}}{4} + \frac{1}{2}\right|^2 = \frac{(1+\cos\omega_0 t)^2}{4}$$

$$|\langle 1,0|_y|\psi(t)\rangle|^2 = \left|\frac{(e^{-i\omega_0 t} - e^{i\omega_0 t})\sqrt{2}}{4}\right|^2 = \frac{\sin^2\omega_0 t}{2}$$

$$|\langle 1,-1|_y|\psi(t)\rangle|^2 = \left|\frac{e^{-i\omega_0 t} + e^{i\omega_0 t}}{4} - \frac{1}{2}\right|^2 = \frac{(1-\cos\omega_0 t)^2}{4}$$

$$\langle S_y\rangle = \frac{(1+\cos\omega_0 t)^2}{4}(\hbar) + 0 + \frac{(1-\cos\omega_0 t)^2}{4}(-\hbar) = \hbar\cos\omega_0 t$$

$$\langle S_z \rangle = \frac{1}{4}(\hbar) + 0 + \frac{1}{4}(-\hbar) = 0$$

Problem 4.12

$$\hat{H} = \omega_0 \hat{S}_x$$

$$|\psi(0)\rangle = |1,1\rangle = \frac{1}{2}|1,1\rangle_x + \frac{\sqrt{2}}{2}|1,0\rangle_x + \frac{1}{2}|1,-1\rangle_x$$

$$|\psi(t)\rangle = \frac{e^{-iE_1 t/\hbar}}{2}|1,1\rangle_x + \frac{\sqrt{2}e^{-iE_0 t/\hbar}}{2}|1,0\rangle_x + \frac{e^{-iE_{-1} t/\hbar}}{2}|1,-1\rangle_x$$

$$= \frac{e^{-i\omega_0 t}}{2}|1,1\rangle_x + \frac{\sqrt{2}}{2}|1,0\rangle_x + \frac{e^{i\omega_0 t}}{2}|1,-1\rangle_x$$

$$\langle 1,-1| = \frac{1}{2}\langle 1,1|_x - \frac{\sqrt{2}}{2}\langle 1,0|_x + \frac{1}{2}\langle 1,-1|_x$$

$$|\langle 1,-1|\psi(t)\rangle|^2 = \left|\frac{e^{-i\omega_0 t} + e^{i\omega_0 t}}{4} - \frac{1}{2}\right|^2 = \frac{(1-\cos\omega_0 t)^2}{4}$$

Problem 4.13

$$\hat{H} \underset{1,2,3}{\longrightarrow} \begin{pmatrix} E_0 & 0 & A \\ 0 & E_1 & 0 \\ A & 0 & E_0 \end{pmatrix}$$

Find eigenstates:

$$\begin{vmatrix} E_0-\lambda & 0 & A \\ 0 & E_1-\lambda & 0 \\ A & 0 & E_0-\lambda \end{vmatrix} = (E_0-\lambda)(E_1-\lambda)(E_0-\lambda) - A^2(E_1-\lambda) = 0$$

$$(\lambda - E_0)^2 - A^2 = 0$$

$\lambda = E_1, E_0 \pm A$

$$\begin{pmatrix} E_0-\lambda & 0 & A \\ 0 & E_1-\lambda & 0 \\ A & 0 & E_0-\lambda \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0$$

$\lambda = E_1$:

$$(E_0 - E_1)a + Ac = 0$$

$$Aa + (E_0 - E_1)c = 0$$

$$a = c = 0, b = 1$$

$$|E_1\rangle = |2\rangle$$

$\lambda = E_0 + A$:

$$-Aa + Ac = 0$$

$$(E_1 - E_0 - A)b = 0$$

$$Aa - Ac = 0$$

$$a = c = \frac{1}{\sqrt{2}}, b = 0$$

$$|E_0 + A\rangle = \frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|3\rangle$$

$\lambda = E_0 - A$:

$$Aa + Ac = 0$$

$$(E_1 - E_0 + A)b = 0$$

$$Aa + Ac = 0$$

$$a = -c = \frac{1}{\sqrt{2}}, b = 0$$

$$|E_0 - A\rangle = \frac{1}{\sqrt{2}}|1\rangle - \frac{1}{\sqrt{2}}|3\rangle$$

a)

$$|\psi(0)\rangle = |2\rangle$$

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = e^{-iE_1t/\hbar}|E_1\rangle$$

b)

$$|\psi(0)\rangle = |3\rangle = \frac{1}{\sqrt{2}}|E_0 + A\rangle - \frac{1}{\sqrt{2}}|E_0 - A\rangle$$

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = \frac{e^{-i(E_0+A)t/\hbar}}{\sqrt{2}}|E_0 + A\rangle - \frac{e^{-i(E_0-A)t/\hbar}}{\sqrt{2}}|E_0 - A\rangle$$

Problem 4.14

$$\hat{H} \underset{x,y}{\rightarrow} \begin{pmatrix} 0 & -iE_0 \\ iE_0 & 0 \end{pmatrix}$$

$$\hat{H} \underset{x,y}{\rightarrow} \frac{\hbar\omega_0}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$|{+\hbar\omega_0/2}\rangle = |{+\mathbf{y}}\rangle = \frac{1}{\sqrt{2}}|{+\mathbf{z}}\rangle + \frac{i}{\sqrt{2}}|{-\mathbf{z}}\rangle$$

$$|{-\hbar\omega_0/2}\rangle = |{-\mathbf{y}}\rangle = \frac{1}{\sqrt{2}}|{+\mathbf{z}}\rangle - \frac{i}{\sqrt{2}}|{-\mathbf{z}}\rangle$$

a) By analogy, the eigenstates and eigenvalues are:

$$|+E_0\rangle = \frac{1}{\sqrt{2}}|x\rangle + \frac{i}{\sqrt{2}}|y\rangle$$
$$|-E_0\rangle = \frac{1}{\sqrt{2}}|x\rangle - \frac{i}{\sqrt{2}}|y\rangle$$

b)

$$|\psi(0)\rangle = |x\rangle = \frac{1}{\sqrt{2}}|+E_0\rangle + \frac{1}{\sqrt{2}}|-E_0\rangle$$
$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle = \frac{e^{-iE_0t/\hbar}}{\sqrt{2}}|+E_0\rangle + \frac{e^{iE_0t/\hbar}}{\sqrt{2}}|-E_0\rangle$$
$$|\langle x|\psi(t)\rangle|^2 = \left|\frac{e^{-iE_0t/\hbar} + e^{iE_0t/\hbar}}{2}\right|^2 = \cos^2\frac{E_0t}{\hbar}$$
$$|y\rangle = -\frac{i}{\sqrt{2}}|+E_0\rangle + \frac{i}{\sqrt{2}}|-E_0\rangle$$
$$|\langle y|\psi(t)\rangle|^2 = \left|\frac{-ie^{-iE_0t/\hbar} + ie^{iE_0t/\hbar}}{2}\right|^2 = \sin^2\frac{E_0t}{\hbar}$$

The photon polarization is oscillating between the x and y states.

Problem 4.15

$$[\hat{A},\hat{B}] = i\hat{C} \to \Delta A\Delta B \geq \frac{|\langle C\rangle|}{2}$$
$$\langle[\hat{A},\hat{B}]\rangle = i\langle\hat{C}\rangle$$
$$|\langle\hat{C}\rangle| = |\langle[\hat{A},\hat{B}]\rangle|$$
$$\frac{d\langle A\rangle}{dt} = \frac{i}{\hbar}\langle\psi(t)|[\hat{H},\hat{A}]\psi(t)\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{A}]\rangle$$
$$|\langle[\hat{H},\hat{A}]\rangle| = \hbar|d\langle A\rangle/dt|$$
$$\Delta E\Delta A \geq \frac{|\langle[\hat{H},\hat{A}]\rangle|}{2}$$
$$\Delta E\left(\frac{\Delta A}{|d\langle A\rangle/dt|}\right) \geq \frac{\hbar}{2}$$

Δt is time needed for expected value of observable to change significantly.

Problem 4.16 - Skipped