Solutions - Chapter 2

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Problem 2.1

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$\left(1+\frac{x}{N}\right)^N = \sum_{k=0}^{N} \binom{N}{k} (1)^{N-k} \left(\frac{x}{N}\right)^k = \sum_{k=0}^{N} \frac{N!}{k!(N-k)!} \frac{x^k}{N^k}$$

$$\lim_{N\to\infty} \frac{N!}{N^k (N-k)!} = 1$$

$$\lim_{N\to\infty} \left(1+\frac{x}{N}\right)^N = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... = e^x$$

Problem 2.2

$$\hat{P}_+ |\lambda\rangle = \lambda |\lambda\rangle$$

$$\hat{P}_+ |\lambda\rangle = \hat{P}_+^2 |\lambda\rangle = \hat{P}_+(\hat{P}_+ |\lambda\rangle) = \hat{P}_+(\lambda |\lambda\rangle) = \lambda \hat{P}_+ |\lambda\rangle$$

$$\hat{P}_+ |\lambda\rangle = 0, \lambda = 0$$
$$\hat{P}_+ |\lambda\rangle \neq 0, \lambda = 1$$

Problem 2.3

$$\hat{R}(\phi k) |+\mathbf{z}\rangle = e^{-i\phi/2} |+\mathbf{z}\rangle$$
$$\hat{R}(\phi k) |-\mathbf{z}\rangle = e^{i\phi/2} |-\mathbf{z}\rangle$$

$$\hat{R}(\phi k) \underset{S_z}{\rightarrow} \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}$$

$$\hat{R}^\dagger(\phi k) = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}$$

$$\begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} \begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$\hat{R}^{\dagger}(\phi k)\hat{R}(\phi k) = 1$$

Problem 2.4

$$|\pm\mathbf{x}\rangle = \frac{1}{\sqrt{2}}|{+\mathbf{z}}\rangle \pm \frac{1}{\sqrt{2}}|{-\mathbf{z}}\rangle$$

$$|\pm\mathbf{y}\rangle = \frac{1}{\sqrt{2}}|{+\mathbf{z}}\rangle \pm \frac{i}{\sqrt{2}}|{-\mathbf{z}}\rangle$$

$$\langle\pm\mathbf{y}| = \frac{1}{\sqrt{2}}\langle{+\mathbf{z}}| \mp \frac{i}{\sqrt{2}}\langle{-\mathbf{z}}|$$

$$\langle{+\mathbf{y}}|+\mathbf{x}\rangle = \langle{-\mathbf{y}}|-\mathbf{x}\rangle = \frac{1-i}{2}$$

$$\langle{-\mathbf{y}}|+\mathbf{x}\rangle = \langle{+\mathbf{y}}|-\mathbf{x}\rangle = \frac{1+i}{2}$$

$$|{+\mathbf{x}}\rangle = |{+\mathbf{y}}\rangle\langle{+\mathbf{y}}|+\mathbf{x}\rangle + |{-\mathbf{y}}\rangle\langle{-\mathbf{y}}|+\mathbf{x}\rangle$$

$$|{-\mathbf{x}}\rangle = |{+\mathbf{y}}\rangle\langle{+\mathbf{y}}|-\mathbf{x}\rangle + |{-\mathbf{y}}\rangle\langle{-\mathbf{y}}|-\mathbf{x}\rangle$$

$$|{+\mathbf{x}}\rangle \underset{S_y}{\rightarrow} \begin{pmatrix} \frac{1-i}{2} \\ \frac{1+i}{2} \end{pmatrix}$$

$$|{-\mathbf{x}}\rangle \underset{S_y}{\rightarrow} \begin{pmatrix} \frac{1+i}{2} \\ \frac{1-i}{2} \end{pmatrix}$$

Problem 2.5

$$\hat{J}_z|\pm\mathbf{z}\rangle = \pm\frac{\hbar}{2}|\pm\mathbf{z}\rangle$$

$$\hat{J}_z \underset{S_z}{\rightarrow} \begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix}$$

$$\underset{S_y}{\hat{J}_z} \rightarrow S^{\dagger}\underset{S_z}{\hat{J}_z}S$$

$$S = \begin{pmatrix} \langle{+\mathbf{z}}|+\mathbf{y}\rangle & \langle{+\mathbf{z}}|-\mathbf{y}\rangle \\ \langle{-\mathbf{z}}|+\mathbf{y}\rangle & \langle{-\mathbf{z}}|-\mathbf{y}\rangle \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$$S^{\dagger}\hat{J}_zS = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} \hbar/2 & 0 \\ 0 & -\hbar/2 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \hbar/2 & i\hbar/2 \\ \hbar/2 & -i\hbar/2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & \hbar \\ \hbar & 0 \end{pmatrix}$$

$$\hat{J}_z \underset{S_y}{\rightarrow} \begin{pmatrix} 0 & \hbar/2 \\ \hbar/2 & 0 \end{pmatrix}$$

$$\langle S_z \rangle = \langle \psi | \hat{J}_z | \psi \rangle$$

$$|\psi\rangle = |-\mathbf{y}\rangle$$

$$\hat{J}_z \, |\pm\mathbf{y}\rangle = \frac{\hbar}{2} \, |\mp\mathbf{y}\rangle$$

$$\langle S_z \rangle = \langle -\mathbf{y} | \hat{J}_z | -\mathbf{y} \rangle = \frac{\hbar}{2} \langle -\mathbf{y} | +\mathbf{y} \rangle = 0$$

Problem 2.6

$$\hat{R}(\theta j) = e^{-i\hat{J}_y \theta/\hbar}$$

$$\hat{J}_y \, |\pm\mathbf{y}\rangle = \pm\frac{\hbar}{2} \, |\pm\mathbf{y}\rangle$$

$$|+\mathbf{z}\rangle = \frac{1}{\sqrt{2}} \, |+\mathbf{y}\rangle + \frac{1}{\sqrt{2}} \, |-\mathbf{y}\rangle$$

$$|-\mathbf{z}\rangle = -\frac{i}{\sqrt{2}} \, |+\mathbf{y}\rangle + \frac{i}{\sqrt{2}} \, |-\mathbf{y}\rangle$$

$$\hat{J}_y \, |+\mathbf{z}\rangle = \frac{1}{\sqrt{2}} \hat{J}_y \, |+\mathbf{y}\rangle + \frac{1}{\sqrt{2}} \hat{J}_y \, |-\mathbf{y}\rangle = \frac{1}{\sqrt{2}} \left(\frac{\hbar}{2}\right) |+\mathbf{y}\rangle + \frac{1}{\sqrt{2}} \left(-\frac{\hbar}{2}\right) |-\mathbf{y}\rangle$$

$$= \frac{1}{\sqrt{2}} \frac{\hbar}{2} \left[\left(\frac{1}{\sqrt{2}} \, |+\mathbf{z}\rangle + \frac{i}{\sqrt{2}} \, |-\mathbf{z}\rangle \right) - \left(\frac{1}{\sqrt{2}} \, |+\mathbf{z}\rangle - \frac{i}{\sqrt{2}} \, |-\mathbf{z}\rangle \right) \right] = \frac{1}{\sqrt{2}} \frac{\hbar}{2} \sqrt{2} i \, |-\mathbf{z}\rangle$$

$$\hat{J}_y \, |-\mathbf{z}\rangle = -\frac{i}{\sqrt{2}} \hat{J}_y \, |+\mathbf{y}\rangle + \frac{i}{\sqrt{2}} \hat{J}_y \, |-\mathbf{y}\rangle = -\frac{i}{\sqrt{2}} \left(\frac{\hbar}{2}\right) |+\mathbf{y}\rangle + \frac{i}{\sqrt{2}} \left(-\frac{\hbar}{2}\right) |-\mathbf{y}\rangle$$

$$= -\frac{i}{\sqrt{2}} \frac{\hbar}{2} \left[\left(\frac{1}{\sqrt{2}} \, |+\mathbf{z}\rangle + \frac{i}{\sqrt{2}} \, |-\mathbf{z}\rangle \right) + \left(\frac{1}{\sqrt{2}} \, |+\mathbf{z}\rangle - \frac{i}{\sqrt{2}} \, |-\mathbf{z}\rangle \right) \right] = -\frac{i}{\sqrt{2}} \frac{\hbar}{2} \sqrt{2} \, |+\mathbf{z}\rangle$$

$$\hat{J}_y \, |\pm\mathbf{z}\rangle = \frac{\pm i\hbar}{2} \, |\mp\mathbf{z}\rangle$$

$$\hat{R}(\theta j) \, |+\mathbf{z}\rangle = e^{-i\hat{J}_y \theta/\hbar} \, |+\mathbf{z}\rangle$$

$$= \left[1 - \frac{i\theta \hat{J}_y}{\hbar} + \frac{1}{2!} \left(-\frac{i\theta \hat{J}_y}{\hbar} \right)^2 + ... \right] |+\mathbf{z}\rangle = |+\mathbf{z}\rangle + \left(-\frac{i\theta}{\hbar} \right) \hat{J}_y \, |+\mathbf{z}\rangle + \frac{1}{2!} \left(-\frac{i\theta}{\hbar} \right)^2 \hat{J}_y^2 \, |+\mathbf{z}\rangle + ...$$

$$\hat{J}_y^2 \, |+\mathbf{z}\rangle = \hat{J}_y (\hat{J}_y \, |+\mathbf{z}\rangle) = \hat{J}_y (\frac{i\hbar}{2} \, |-\mathbf{z}\rangle) = \left(\frac{\hbar}{2}\right)^2 |+\mathbf{z}\rangle$$

$$\hat{J}_y^4 \, |+\mathbf{z}\rangle = \hat{J}_y^2 (\hat{J}_y^2 \, |+\mathbf{z}\rangle) = \hat{J}_y^2 (\left(\frac{\hbar}{2}\right)^2 |+\mathbf{z}\rangle) = \left(\frac{\hbar}{2}\right)^4 |+\mathbf{z}\rangle$$

$$\hat{J}_y^{2n} \, |+\mathbf{z}\rangle = \left(\frac{\hbar}{2}\right)^{2n} |+\mathbf{z}\rangle$$

$$\hat{J}_y^3 |+\mathbf{z}\rangle = \hat{J}_y(\hat{J}_y^2 |+\mathbf{z}\rangle) = \hat{J}_y(\left(\frac{\hbar}{2}\right)^2 |+\mathbf{z}\rangle) = i \left(\frac{\hbar}{2}\right)^3 |-\mathbf{z}\rangle$$

$$\hat{J}_y^5 |+\mathbf{z}\rangle = \hat{J}_y^2(\hat{J}_y^3 |+\mathbf{z}\rangle) = \hat{J}_y^2(i \left(\frac{\hbar}{2}\right)^3 |-\mathbf{z}\rangle) = i \left(\frac{\hbar}{2}\right)^5 |-\mathbf{z}\rangle$$

$$\hat{J}_y^{2n+1} |+\mathbf{z}\rangle = i \left(\frac{\hbar}{2}\right)^{2n+1} |+\mathbf{z}\rangle$$

$$\frac{1}{(2n)!} \left(-\frac{i\theta}{\hbar}\right)^{2n} \hat{J}_y^{2n} |+\mathbf{z}\rangle = \frac{(-1)^n}{(2n)!} \left(\frac{\theta}{2}\right)^{2n} |+\mathbf{z}\rangle$$

$$\frac{1}{(2n+1)!} \left(-\frac{i\theta}{\hbar}\right)^{2n+1} \hat{J}_y^{2n+1} |+\mathbf{z}\rangle = \frac{(-1)^n}{(2n+1)!} \left(\frac{\theta}{2}\right)^{2n+1} |+\mathbf{z}\rangle$$

$$\sum \frac{(-1)^n}{(2n)!} \left(\frac{\theta}{2}\right)^{2n} = \cos\frac{\theta}{2}$$

$$\sum \frac{(-1)^n}{(2n+1)!} \left(\frac{\theta}{2}\right)^{2n+1} = \sin\frac{\theta}{2}$$

$$\hat{R}(\theta j) |+\mathbf{z}\rangle = \cos\frac{\theta}{2} |+\mathbf{z}\rangle + \sin\frac{\theta}{2} |-\mathbf{z}\rangle$$

$$\hat{R}(\frac{\pi}{2} j) |+\mathbf{z}\rangle = \cos\frac{\pi}{4} |+\mathbf{z}\rangle + \sin\frac{\pi}{4} |-\mathbf{z}\rangle = |+\mathbf{x}\rangle$$

Problem 2.7

$$\hat{P}_+ |+\mathbf{z}\rangle = |+\mathbf{z}\rangle$$

$$\hat{P}_+ |-\mathbf{z}\rangle = 0$$

$$\hat{P}_- |+\mathbf{z}\rangle = 0$$

$$\hat{P}_- |-\mathbf{z}\rangle = |-\mathbf{z}\rangle$$

$$\hat{P}_+ \underset{S_z}{\rightarrow} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\hat{P}_- \underset{S_z}{\rightarrow} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\underset{S_y}{\hat{P}_+} \rightarrow S^\dagger \underset{S_z}{\hat{P}_+} S$$

$$\underset{S_y}{\hat{P}_-} \rightarrow S^\dagger \underset{S_z}{\hat{P}_-} S$$

$$S = \begin{pmatrix} \langle +\mathbf{z}| + \mathbf{y}\rangle & \langle +\mathbf{z}| - \mathbf{y}\rangle \\ \langle -\mathbf{z}| + \mathbf{y}\rangle & \langle -\mathbf{z}| - \mathbf{y}\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$$S^\dagger \hat{P}_+ S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$$\hat{P}_+ \underset{S_y}{\rightarrow} \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$S^\dagger \hat{P}_- S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ 0 & i \end{pmatrix}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$$\hat{P}_- \underset{S_y}{\rightarrow} \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

$$\hat{P}_+^2 \rightarrow \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rightarrow \hat{P}_+$$

$$\hat{P}_-^2 \rightarrow \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \rightarrow \hat{P}_-$$

$$\hat{P}_+\hat{P}_- \rightarrow \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = 0$$

$$\hat{P}_-\hat{P}_+ \rightarrow \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0$$

Problem 2.8

$$|\psi\rangle = \sqrt{\frac{2}{3}}\,|x\rangle + \frac{i}{\sqrt{3}}\,|y\rangle$$

a)

$$|\langle y|\psi\rangle|^2 = \left|\frac{i}{\sqrt{3}}\right|^2 = \frac{1}{3}$$

$$|y'\rangle = -\sin\phi\,|x\rangle + \cos\phi\,|y\rangle$$

$$|\langle y'|\psi\rangle|^2 = \left|(-\sin\phi\,\langle x| + \cos\phi\,\langle y|)\left(\sqrt{\frac{2}{3}}\,|x\rangle + \frac{i}{\sqrt{3}}\,|y\rangle\right)\right|^2 = \left|-\sin\phi\sqrt{\frac{2}{3}} + \cos\phi\frac{i}{\sqrt{3}}\right|^2$$

$$= \left(-\sin\phi\sqrt{\frac{2}{3}}\right)^2 + \left(\cos\phi\frac{1}{\sqrt{3}}\right)^2 = \frac{2}{3}\sin^2\phi + \frac{1}{3}\cos^2\phi$$

c)

$$|R\rangle = \frac{1}{\sqrt{2}}(|x\rangle + i\,|y\rangle)$$

$$|L\rangle = \frac{1}{\sqrt{2}}(|x\rangle - i\,|y\rangle)$$

$$|x\rangle = \frac{1}{\sqrt{2}}\,|R\rangle + \frac{1}{\sqrt{2}}\,|L\rangle$$

$$|y\rangle = -\frac{i}{\sqrt{2}}\,|R\rangle + \frac{i}{\sqrt{2}}\,|L\rangle$$

$$|\psi\rangle = \sqrt{\frac{2}{3}}\left(\frac{1}{\sqrt{2}}|R\rangle + \frac{1}{\sqrt{2}}|L\rangle\right) + \frac{i}{\sqrt{3}}\left(-\frac{i}{\sqrt{2}}|R\rangle + \frac{i}{\sqrt{2}}|L\rangle\right)$$

$$|\psi\rangle = \left(\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{6}}\right)|R\rangle + \left(\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{6}}\right)|L\rangle$$

Angular momentum
$|R\rangle : +\hbar$
$|L\rangle : -\hbar$

$$\langle\psi|\hat{J}_z|\psi\rangle = |\langle R|\psi\rangle|^2(+\hbar) + |\langle L|\psi\rangle|^2(-\hbar) = \frac{4}{3\sqrt{2}}\hbar$$

Clockwise:

$$\tau = \frac{dL}{dt} = \frac{4N\hbar}{3\sqrt{2}}$$

d)
a: same
b:

$$|\langle y'|\psi\rangle|^2 = \left(-\sin\phi\sqrt{\frac{2}{3}} + \cos\phi\frac{1}{\sqrt{3}}\right)^2$$

c:

$$|\psi\rangle = \left(\frac{1}{\sqrt{3}} - \frac{i}{\sqrt{6}}\right)|R\rangle + \left(\frac{1}{\sqrt{3}} + \frac{i}{\sqrt{6}}\right)|L\rangle$$

$$|\langle R|\psi\rangle|^2 = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

$$|\langle L|\psi\rangle|^2 = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

$$\tau = 0$$

Problem 2.9

a)

$$|y'\rangle = -\sin\phi\,|x\rangle + \cos\phi\,|y\rangle$$

$$|y_1\rangle = -\sin\frac{\phi}{N}|x\rangle + \cos\frac{\phi}{N}|y\rangle$$

$$|y_2\rangle = -\sin\frac{\phi}{N}|x_1\rangle + \cos\frac{\phi}{N}|y_1\rangle$$

$$|y_N\rangle = -\sin\frac{\phi}{N}|x_{N-1}\rangle + \cos\frac{\phi}{N}|y_{N-1}\rangle$$

$$P_1 = |\langle y_1|y\rangle|^2 = \cos^2\frac{\phi}{N}$$

$$P_2 = |\langle y_2|y_1\rangle|^2 = \cos^2\frac{\phi}{N}$$

$$P_N = |\langle y_N|y_{N-1}\rangle|^2 = \cos^2\frac{\phi}{N}$$

$$P_{trans} = P_1P_2...P_N = \left(\cos^2\frac{\phi}{N}\right)^N = \cos^{2N}\frac{\phi}{N}$$

b)

$$\lim_{N\to\infty}\cos^{2N}\frac{\phi}{N} = 1$$

c)

$$\phi = \frac{\pi}{2}$$

$$P_{trans} = \cos^{2N}\frac{\pi}{2N}$$

Performing a measurement changes the polarization state of the photon.

Problem 2.10

a)

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\begin{pmatrix} |x\rangle \\ |y\rangle \end{pmatrix} = \begin{pmatrix} |R\rangle \\ |L\rangle \end{pmatrix}$$

$$S \to \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

b)

$$S^\dagger S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} = I$$

Problem 2.11

$$\hat{J}_z|R\rangle = \hbar|R\rangle$$

$$\hat{J}_z|L\rangle = -\hbar|L\rangle$$

$$\hat{J}_z \underset{R,L}{\to} \begin{pmatrix} \hbar & 0 \\ 0 & -\hbar \end{pmatrix}$$

$$\underset{x,y}{\hat{J}_z} = S^\dagger \underset{R,L}{\hat{J}_z} S$$

$$S = \begin{pmatrix} \langle R|x\rangle & \langle R|y\rangle \\ \langle L|x\rangle & \langle L|y\rangle \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

$$S^\dagger\hat{J}_zS = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}\begin{pmatrix} \hbar & 0 \\ 0 & -\hbar \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} \hbar & -\hbar \\ i\hbar & i\hbar \end{pmatrix}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & -2i\hbar \\ 2i\hbar & 0 \end{pmatrix}$$

$$\hat{J}_z \underset{x,y}{\to} \begin{pmatrix} 0 & -i\hbar \\ i\hbar & 0 \end{pmatrix}$$

$$\hat{J}_z^\dagger = \hat{J}_z$$

Problem 2.12

$$|\psi\rangle = a\,|R\rangle + b\,|L\rangle$$

$$\langle S_z\rangle = \langle\psi|\hat{J}_z|\psi\rangle$$

$$\hat{J}_z\,|\psi\rangle \rightarrow \begin{pmatrix} \hbar & 0 \\ 0 & -\hbar \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \hbar a \\ -\hbar b \end{pmatrix}$$

$$\langle\psi|\hat{J}_z|\psi\rangle = \begin{pmatrix} \text{a*} & \text{b*} \end{pmatrix} \begin{pmatrix} \hbar a \\ -\hbar b \end{pmatrix} = \hbar a^2 - \hbar b^2$$

$$\langle S_z\rangle = \hbar(a^2 - b^2)$$

Problem 2.13

$$\hat{R}(\phi k) \underset{x,y}{\rightarrow} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

$$|R\rangle \underset{x,y}{\rightarrow} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$|L\rangle \underset{x,y}{\rightarrow} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\hat{R}(\phi k)\,|R\rangle = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\phi - i\sin\phi \\ \sin\phi + i\cos\phi \end{pmatrix} = e^{-i\phi}\,|R\rangle$$

$$\hat{R}(\phi k)\,|L\rangle = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\phi + i\sin\phi \\ \sin\phi - i\cos\phi \end{pmatrix} = e^{i\phi}\,|L\rangle$$

Problem 2.14

$$\hat{P}_x = |x\rangle\,\langle x|$$

$$\hat{P}_y = |y\rangle\,\langle y|$$

$$\hat{P}_x^2 = |x\rangle\,\langle x|x\rangle\,\langle x| = |x\rangle\,\langle x| = \hat{P}_x$$

$$\hat{P}_y^2 = |y\rangle\,\langle y|y\rangle\,\langle y| = |y\rangle\,\langle y| = \hat{P}_y$$

$$\hat{P}_x\hat{P}_y = |x\rangle\,\langle x|y\rangle\,\langle y| = 0$$

$$\hat{P}_y\hat{P}_x = |y\rangle\,\langle y|x\rangle\,\langle x| = 0$$

Problem 2.15

$$\hat{J}_z\,|R\rangle = \hbar\,|R\rangle$$

$$\hat{J}_z \left|L\right\rangle = -\hbar \left|L\right\rangle$$

$$\hat{J}_z = \hbar \left|R\right\rangle \left\langle R\right| - \hbar \left|L\right\rangle \left\langle L\right|$$

Problem 2.16

$$\left\langle x'\right| = \cos\phi \left\langle x\right| + \sin\phi \left\langle y\right|$$

$$|\left\langle x'|R\right\rangle|^2 = \left|\frac{1}{\sqrt{2}}\cos\phi + \frac{i}{\sqrt{2}}\sin\phi\right|^2 = \frac{1}{2}\cos^2\phi + \frac{1}{2}\sin^2\phi = \frac{1}{2}$$

Problem 2.17 - Skipped

Problem 2.18 - Skipped

Problem 2.19

$\left|a_n\right\rangle$ - orthonormal basis

$$\left\langle a_i|a_j\right\rangle = \delta_{ij}$$

$$\hat{U}^\dagger \hat{U} = 1$$

$$\left\langle a_i|\hat{U}^\dagger \hat{U}|a_j\right\rangle = \left\langle a_i|a_j\right\rangle = \delta_{ij}$$

$\hat{U}\left|a_n\right\rangle$ - orthonormal basis