Solutions - Chapter 1

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Problem 1.1 - Skipped

Problem 1.2 - Skipped

Problem 1.3

a)

$$|+\mathbf{n}\rangle = \cos\frac{\theta}{2}|+\mathbf{z}\rangle + e^{i\phi}\sin\frac{\theta}{2}|-\mathbf{z}\rangle$$

$$\theta = \frac{\pi}{2}, \phi = 0 \rightarrow |+\mathbf{n}\rangle = \cos\frac{\pi}{4}|+\mathbf{z}\rangle + e^{0i}\sin\frac{\pi}{4}|-\mathbf{z}\rangle = \frac{1}{\sqrt{2}}|+\mathbf{z}\rangle + \frac{1}{\sqrt{2}}|-\mathbf{z}\rangle = |+\mathbf{x}\rangle$$

$$\theta = \frac{\pi}{2}, \phi = \frac{\pi}{2} \rightarrow |+\mathbf{n}\rangle = \cos\frac{\pi}{4}|+\mathbf{z}\rangle + e^{\frac{i\pi}{2}}\sin\frac{\pi}{4}|-\mathbf{z}\rangle = \frac{1}{\sqrt{2}}|+\mathbf{z}\rangle + \frac{i}{\sqrt{2}}|-\mathbf{z}\rangle = |+\mathbf{y}\rangle$$

b)

$$\langle +\mathbf{z}| + \mathbf{n}\rangle = \cos\frac{\theta}{2}\langle +\mathbf{z}| + \mathbf{z}\rangle + e^{i\phi}\sin\frac{\theta}{2}\langle +\mathbf{z}| - \mathbf{z}\rangle = \cos\frac{\theta}{2}$$

$$P\left(S_z = \frac{\hbar}{2}\right) = |\langle +\mathbf{z}| + \mathbf{n}\rangle|^2 = \cos^2\frac{\theta}{2}$$

$$\langle -\mathbf{z}| + \mathbf{n}\rangle = \cos\frac{\theta}{2}\langle -\mathbf{z}| + \mathbf{z}\rangle + e^{i\phi}\sin\frac{\theta}{2}\langle -\mathbf{z}| - \mathbf{z}\rangle = e^{i\phi}\sin\frac{\theta}{2}$$

$$P\left(S_z = -\frac{\hbar}{2}\right) = |\langle -\mathbf{z}| + \mathbf{n}\rangle|^2 = \sin^2\frac{\theta}{2}$$

c)

$$\langle S_z\rangle = \cos^2\frac{\theta}{2}\left(\frac{\hbar}{2}\right) + \sin^2\frac{\theta}{2}\left(-\frac{\hbar}{2}\right) = \cos\theta\left(\frac{\hbar}{2}\right)$$

$$\langle S_z\rangle^2 = \cos^2\theta\left(\frac{\hbar}{2}\right)^2$$

$$\langle {S_z}^2\rangle = \cos^2\frac{\theta}{2}\left(\frac{\hbar}{2}\right)^2 + \sin^2\frac{\theta}{2}\left(-\frac{\hbar}{2}\right)^2 = \left(\frac{\hbar}{2}\right)^2$$

$$\Delta S_z = \sqrt{\langle {S_z}^2\rangle - \langle S_z\rangle^2} = \sqrt{\left(\frac{\hbar}{2}\right)^2 - \cos^2\theta\left(\frac{\hbar}{2}\right)^2} = \sin\theta\left(\frac{\hbar}{2}\right)$$

Problem 1.4

b)

$$|+\mathbf{x}\rangle = \frac{1}{\sqrt{2}}\,|+\mathbf{z}\rangle + \frac{1}{\sqrt{2}}\,|-\mathbf{z}\rangle$$

$$\langle +\mathbf{x}| + \mathbf{z}\rangle = \langle +\mathbf{z}| + \mathbf{x}\rangle^* = \frac{1}{\sqrt{2}}$$

$$\langle +\mathbf{x}| - \mathbf{z}\rangle = \langle -\mathbf{z}| + \mathbf{x}\rangle^* = \frac{1}{\sqrt{2}}$$

$$\langle +\mathbf{x}| + \mathbf{n}\rangle = \cos\frac{\theta}{2}\,\langle +\mathbf{x}| + \mathbf{z}\rangle + e^{i\phi}\sin\frac{\theta}{2}\,\langle +\mathbf{x}| - \mathbf{z}\rangle = \frac{1}{\sqrt{2}}\left(\cos\frac{\theta}{2} + e^{i\phi}\sin\frac{\theta}{2}\right)$$

$$|\,\langle +\mathbf{x}| + \mathbf{n}\rangle\,| = \frac{1}{\sqrt{2}}\left|\left[\cos\frac{\theta}{2} + (\cos\phi + i\sin\phi)\sin\frac{\theta}{2}\right]\right|$$

$$= \frac{1}{\sqrt{2}}\sqrt{\left(\cos\frac{\theta}{2} + \cos\phi\sin\frac{\theta}{2}\right)^2 + \left(\sin\phi\sin\frac{\theta}{2}\right)^2}$$

$$= \frac{1}{\sqrt{2}}\sqrt{\cos^2\frac{\theta}{2} + 2\cos\phi\cos\frac{\theta}{2}\sin\frac{\theta}{2} + \cos^2\phi\sin^2\frac{\theta}{2} + \sin^2\phi\sin^2\frac{\theta}{2}}$$

$$= \frac{1}{\sqrt{2}}\sqrt{1 + \cos\phi\sin\theta}$$

$$P\left(S_x = \frac{\hbar}{2}\right) = |\,\langle +\mathbf{x}| + \mathbf{n}\rangle\,|^2 = \frac{1 + \cos\phi\sin\theta}{2}$$

$$P\left(S_x = -\frac{\hbar}{2}\right) = 1 - P\left(S_x = \frac{\hbar}{2}\right) = 1 - \frac{1 + \cos\phi\sin\theta}{2} = \frac{1 - \cos\phi\sin\theta}{2}$$

c)

$$\langle S_x\rangle = \frac{1 + \cos\phi\sin\theta}{2}\left(\frac{\hbar}{2}\right) + \frac{1 - \cos\phi\sin\theta}{2}\left(-\frac{\hbar}{2}\right) = \cos\phi\sin\theta\left(\frac{\hbar}{2}\right)$$

$$\langle S_x\rangle^2 = \cos^2\phi\sin^2\theta\left(\frac{\hbar}{2}\right)^2$$

$$\langle {S_x}^2\rangle = \frac{1 + \cos\phi\sin\theta}{2}\left(\frac{\hbar}{2}\right)^2 + \frac{1 - \cos\phi\sin\theta}{2}\left(-\frac{\hbar}{2}\right)^2 = \left(\frac{\hbar}{2}\right)^2$$

$$\Delta S_x = \sqrt{\langle {S_x}^2\rangle - \langle S_x\rangle^2} = \sqrt{1 - \cos^2\phi\sin^2\theta}\left(\frac{\hbar}{2}\right)$$

Problem 1.5

a)

$$|+\mathbf{y}\rangle = \frac{1}{\sqrt{2}}|+\mathbf{z}\rangle + \frac{i}{\sqrt{2}}|-\mathbf{z}\rangle$$

$$\langle +\mathbf{y}| + \mathbf{z}\rangle = \langle +\mathbf{z}| + \mathbf{y}\rangle^* = \frac{1}{\sqrt{2}}$$

$$\langle +\mathbf{y}| - \mathbf{z}\rangle = \langle -\mathbf{z}| + \mathbf{y}\rangle^* = -\frac{i}{\sqrt{2}}$$

$$\langle +\mathbf{y}|+\mathbf{n}\rangle = \cos\frac{\theta}{2}\langle +\mathbf{y}| + \mathbf{z}\rangle + e^{i\phi}\sin\frac{\theta}{2}\langle +\mathbf{y}| - \mathbf{z}\rangle = \frac{1}{\sqrt{2}}\left(\cos\frac{\theta}{2} - ie^{i\phi}\sin\frac{\theta}{2}\right)$$

$$|\langle +\mathbf{y}|+\mathbf{n}\rangle| = \frac{1}{\sqrt{2}}\left|\cos\frac{\theta}{2} - i(\cos\phi + i\sin\phi)\sin\frac{\theta}{2}\right|$$

$$= \frac{1}{\sqrt{2}}\sqrt{\left(\cos\frac{\theta}{2} + \sin\phi\sin\frac{\theta}{2}\right)^2 + \left(\cos\phi\sin\frac{\theta}{2}\right)^2}$$

$$= \frac{1}{\sqrt{2}}\sqrt{\cos^2\frac{\theta}{2} + 2\sin\phi\cos\frac{\theta}{2}\sin\frac{\theta}{2} + \sin^2\phi\sin^2\frac{\theta}{2} + \cos^2\phi\sin^2\frac{\theta}{2}}$$

$$= \frac{1}{\sqrt{2}}\sqrt{1 + \sin\phi\sin\theta}$$

$$P\left(S_y = \frac{\hbar}{2}\right) = |\langle +\mathbf{y}| + \mathbf{n}\rangle|^2 = \frac{1 + \sin\phi\sin\theta}{2}$$

b)

$$\langle +\mathbf{n}| + \mathbf{y}\rangle = \langle +\mathbf{y}| + \mathbf{n}\rangle^* = \frac{1}{\sqrt{2}}\left(\cos\frac{\theta}{2} + ie^{-i\phi}\sin\frac{\theta}{2}\right)$$

$$|\langle +\mathbf{n}| + \mathbf{y}\rangle|^2 = |\langle +\mathbf{y}| + \mathbf{n}\rangle|^2 = \frac{1 + \sin\phi\sin\theta}{2}$$

Problem 1.6

$$|-\mathbf{n}\rangle = \sin\frac{\theta}{2}|+\mathbf{z}\rangle - e^{i\phi}\cos\frac{\theta}{2}|-\mathbf{z}\rangle$$

$$\langle -\mathbf{n}| = \sin\frac{\theta}{2}\langle +\mathbf{z}| - e^{-i\phi}\cos\frac{\theta}{2}\langle -\mathbf{z}|$$

$$\langle +\mathbf{n}| = \cos\frac{\theta}{2}\langle +\mathbf{z}| + e^{-i\phi}\sin\frac{\theta}{2}\langle -\mathbf{z}|$$

$$\langle +\mathbf{n}| - \mathbf{n}\rangle = \left(\cos\frac{\theta}{2}\langle +\mathbf{z}| + e^{-i\phi}\sin\frac{\theta}{2}\langle -\mathbf{z}|\right)\left(\sin\frac{\theta}{2}|+\mathbf{z}\rangle - e^{i\phi}\cos\frac{\theta}{2}|-\mathbf{z}\rangle\right)$$

$$= \cos\frac{\theta}{2}\sin\frac{\theta}{2} - \sin\frac{\theta}{2}\cos\frac{\theta}{2} = 0$$

$$\langle -\mathbf{n} | -\mathbf{n} \rangle = \left(\sin\frac{\theta}{2} \langle +\mathbf{z}| - e^{-i\phi}\cos\frac{\theta}{2}\langle -\mathbf{z}| \right) \left(\sin\frac{\theta}{2} |+\mathbf{z}\rangle - e^{i\phi}\cos\frac{\theta}{2}|-\mathbf{z}\rangle \right)$$

$$= \sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} = 1$$

Problem 1.7

a)

$$|+\mathbf{z}\rangle = |+\mathbf{n}\rangle\langle +\mathbf{n}|+\mathbf{z}\rangle + |-\mathbf{n}\rangle\langle -\mathbf{n}|+\mathbf{z}\rangle$$

$$|+\mathbf{z}\rangle = \cos\frac{\theta}{2}|+\mathbf{n}\rangle + \sin\frac{\theta}{2}|-\mathbf{n}\rangle$$

$$\langle +\mathbf{n}|+\mathbf{z}\rangle = \langle +\mathbf{z}|+\mathbf{n}\rangle^* = \cos\frac{\theta}{2}$$

$$\langle -\mathbf{n}|+\mathbf{z}\rangle = \langle +\mathbf{z}|-\mathbf{n}\rangle^* = \sin\frac{\theta}{2}$$

$$P(1\to 2) = \cos^2\frac{\theta}{2}$$

$$|+\mathbf{n}\rangle = |+\mathbf{z}\rangle\langle +\mathbf{z}|+\mathbf{n}\rangle + |-\mathbf{z}\rangle\langle -\mathbf{z}|+\mathbf{n}\rangle$$

$$\phi = 0 \to |+\mathbf{n}\rangle = \cos\frac{\theta}{2}|+\mathbf{z}\rangle + \sin\frac{\theta}{2}|-\mathbf{z}\rangle$$

$$P(2\to 3) = \sin^2\frac{\theta}{2}$$

$$P(1\to 3) = P(1\to 2)P(2\to 3) = \cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2} = \frac{1}{4}\sin^2\theta$$

b)

$$\theta = \frac{\pi}{2} \to P(1\to 3) = \frac{1}{4}\sin^2\frac{\pi}{2} = \frac{1}{4}$$

c)

$$\langle -\mathbf{z}|+\mathbf{z}\rangle = 0$$

$$P(1\to 3) = 0$$

Problem 1.8

$$|\psi\rangle = \frac{i}{\sqrt{3}}|+\mathbf{z}\rangle + \sqrt{\frac{2}{3}}|-\mathbf{z}\rangle$$

$$\langle S_z \rangle = \left|\frac{i}{\sqrt{3}}\right|^2 \left(\frac{\hbar}{2}\right) + \left(\sqrt{\frac{2}{3}}\right)^2 \left(-\frac{\hbar}{2}\right) = -\frac{1}{3}\left(\frac{\hbar}{2}\right)$$

$$\langle S_z \rangle^2 = \frac{1}{9}\left(\frac{\hbar}{2}\right)^2$$

$$\langle {S_z}^2 \rangle = \left|\frac{i}{\sqrt{3}}\right|^2 \left(\frac{\hbar}{2}\right)^2 + \left(\sqrt{\frac{2}{3}}\right)^2 \left(-\frac{\hbar}{2}\right)^2 = \left(\frac{\hbar}{2}\right)^2$$

$$\Delta S_z = \sqrt{\left(\frac{\hbar}{2}\right)^2 - \frac{1}{9}\left(\frac{\hbar}{2}\right)^2} = \frac{2\sqrt{2}}{3}\left(\frac{\hbar}{2}\right)$$

Problem 1.9

$$P(a_i) = |e^{i\delta} \langle a_i|\psi\rangle |^2 = | \langle a_i|\psi\rangle |^2$$

$$\langle S_z \rangle = \sum_i |e^{i\delta} \langle a_i|\psi\rangle |^2 S_i = \sum_i | \langle a_i|\psi\rangle |^2 S_i$$